NEW ASPECTS ON KOENIG'S THEOREM FOR ANGULAR MOMENTUM

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Abstract. *The paper presents the study of angular momentum variation in relation to point of reduction. Thus, it is demonstrated the formula with which one can determine the angular momentum of the rigid body relatively to any point of its if we know the angular momentum of the rigid body relatively to a random point. Based on this formula are deduced Koenig's theorem for angular momentum and as a consequence the mathematical expression of Steiner's theorem.*

Keywords: *angular momentum, point of reduction, mass center*

1. INTRODUCTION

Let us consider a solid rigid body as is shown in figure 1. This solid rigid body is supposed to be in general motion. Let us choose a rectangular Cartesian system T(Ox y z) in space which we conditionally regard as being fixed. The position of a rigid body in space can be specified by the positions of its three points not lying in one straight line [1, 2, 3]. To this end, let us take a coordinate system $T_1(O_1x_1y_1z_1)$ thought of being rigidly connected with the rigid body. The position of the coordinate system $T_1(O_1x_1y_1z_1)$ determines that of the body itself to the fixed reference frame T(O x y z) [6 ,7, 8].

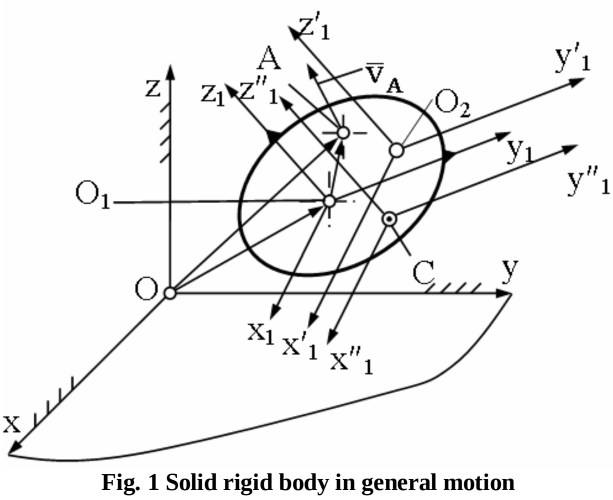

Fig. 1 Solid rigid body in general motion

It will also consider two other reference systems $T_2(C,x''_1,y''_1,z''_1)$ and $T_3(O_2, x'_1, y'_1, z'_1)$ which are also thought as being rigidly connected with the solid rigid body. The origin of the reference frame $T_2(C,x''_1,y''_1,z''_1)$ is a specific point of the body namely its mass center C. Point O_2 is an arbitrary point of the rigid body. Point A is also an arbitrary point which belongs to the rigid body. The axes of the reference systems $T_1(O_1x_1y_1z_1)$, $T_2(C,x''_1,y''_1,z''_1)$ and $T_3(O_2x'_1\ y'_1\ z'_1)$ are considered to be parallel to each other.

We propose to calculate the angular momentum of this rigid solid body about the point "O_1" and study its variation when pole position is changing.

2. ANGULAR MOMENTUM DETERMINATION

The angular momentum of the rigid solid body about the point "O_1" may be calculated with the following relationship [3, 4, 5, 9, 10 -15]:

$$\{K_{O_1}\} = \int_{(C)} [O_1A]\{v_A\}dm \qquad (1)$$

where:

(C) – represents the area occupied by the rigid solid body

$\{v_A\}$ – the velocity vector of the current point "A" of the rigid body

$[O_1A]$ – anti-symmetric matrix associated to the vector $\{O_1A\}$

$\{O_1A\}$ – the position vector of the current point "A" of the solid rigid body

The mathematical expression of the vector $\{O_1A\}$ is the following:

$$\{O_1A\} = [x_A \mid y_A \mid z_A]^T \qquad (2)$$

The mathematical expression of the matrix $[O_1A]$ is the following [4]:

$$[O_1A] = \begin{bmatrix} 0 & -z_A & y_A \\ z_A & 0 & -x_A \\ -y_A & x_A & 0 \end{bmatrix} \qquad (3)$$

In relationship (2) x_A, y_A, z_A represent the coordinates of the point "A" relatively to the body fixed reference frame T_1 $(O_1x_1y_1z_1)$.

3. DETERMINATION OF RIGID BODY ANGULAR MOMENTUM VARIATION WHEN POLE POSITION IS CHANGING

The relation (1) may be written in the following form:

$$\{K_{O_1}\} = \int_{(C)} ([O_1O_2] + [O_2A])\{v_A\}dm \qquad (4)$$

or:

$$\{K_{O_1}\} = [O_1O_2]\int_{(C)}\{v_A\}dm + \int_{(C)}[O_2A]\{v_A\}dm \quad (5)$$

In relation (3) $[O_1O_2]$ and $[O_2A]$ represent the anti-symmetric matrices associated to the vectors $\{O_1O_2\}$ and $\{O_2A\}$ respectively. They have the followings expressions:

$$[O_1O_2] = \begin{bmatrix} 0 & -z_{O_2} & y_{O_2} \\ z_{O_2} & 0 & -x_{O_2} \\ -y_{O_2} & x_{O_2} & 0 \end{bmatrix} \quad (6)$$

$$[O_2A] = \begin{bmatrix} 0 & -p_{z_1} & p_{y_1} \\ p_{z_1} & 0 & -p_{x_1} \\ -p_{y_1} & p_{x_1} & 0 \end{bmatrix} \quad (7)$$

$$p_{x_1} = x_A - x_{O_2} \quad (8)$$

$$p_{y_1} = y_A - y_{O_2} \quad (9)$$

$$p_{z_1} = z_A - z_{O_2} \quad (10)$$

In relations (6)-(10), x_A, y_A, z_A, x_{O_2}, y_{O_2}, z_{O_2} represent the coordinates of points A and O_2 in relation to the body fixed reference frame $T_1(O_1x_1y_1z_1)$.

In relation (5) may be recognized the following quantities:

$$\{H\} = \int_{(C)}\{v_A\}dm = [M]\{v_C\} \quad (11)$$

$$\{K_{O_2}\} = \int_{(C)}[O_2A]\{v_A\}dm \quad (12)$$

The vector quantity expressed by relation (11) represents the momentum of the solid rigid body in general motion. The vector quantity expressed by relation (12) represents the angular momentum of the solid rigid body about the point "O_2".

Using relations (11) and (12) the relationship (5) may be written in the following way:

$$\{K_{O_1}\} = \{K_{O_2}\} + [O_1O_2]\{H\} \quad (13)$$

Relationship (13) represents the mathematical expression of angular momentum variation when pole position is changing.

4. KOENIG'S THEOREME DEDUCTION FOR ANGULAR MOMENTUM

In this chapter we intend to determine the variation of angular momentum of the solid rigid body when pole position changes namely when the new position of the pole coincides with that of the mass center. For this purpose we will use the relationship (13). We will obtain the following result:

$$\{K_{O_1}\} = \{K_C\} + [O_1C]\{H\} \quad (14)$$

Using the relation (11) the relation (14) becomes:

$$\{K_{O_1}\} = \{K_C\} + [O_1C][M]\{v_C\} \quad (15)$$

In relationship (15), $[O_1C]$ represents the anti-symmetric matrix associated to the vector $\{O_1C\}$ and it has the following expression:

$$[O_1C] = \begin{bmatrix} 0 & -\varsigma_1 & \eta_1 \\ \varsigma_1 & 0 & -\xi_1 \\ -\eta_1 & \xi_1 & 0 \end{bmatrix} \quad (16)$$

The vector quantity $\{O_1C\}$ has the following expression:

$$\{O_1C\} = [\xi_1 \;\vdots\; \eta_1 \;\vdots\; \varsigma_1]^T \quad (17)$$

In relationships (16) and (17) the scalar quantities ξ_1, η_1, ζ_1 represent the mass center coordinates relatively to the body fixed reference frame T_1 $(O_1x_1y_1z_1)$.

Relationship (15) represents the mathematical expression of Koenig's theorem for angular momentum.

In this expression the matrix [M] may be written as followings:

$$[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \quad (18)$$

where "m" represents the mass of the solid rigid body.

5. DEDUCTION OF STEINER RELATIONSHIP USING KOENIG'S THEOREM

In this chapter we intend to deduce Steiner's relationship starting from Koenig's theorem. For this purpose we will first calculate the angular momentum of the solid rigid body about its mass center. We will obtain the following relationship:

$$\{K_C\} = \int_{(C)}[CA]\{v_A\}dm = [J_C]\{\omega_1\} \quad (19)$$

$$\{K_C\} = \int_{(C)}[CA]\{v_A\}dm = [D]\{v_1\} \quad (20)$$

$$[J_C] = \begin{bmatrix} J_{x_1''} & -J_{x_1''y_1''} & -J_{x_1''z_1''} \\ -J_{x_1''y_1''} & J_{y_1''} & -J_{y_1''z_1''} \\ -J_{x_1''z_1''} & -J_{y_1''z_1''} & J_{z_1''} \end{bmatrix} \quad (21)$$

Relationship (19) may be written under the following form:

$$\{K_C\} = \int_{(C)} [CA]\{v_A\}dm = [D]\{v_1\} \qquad (22)$$

In relationship (21) scalar quantities $J_{x_1''}, J_{y_1''}, J_{z_1''}$ represent the axial moments of inertia relatively to the axes of the body fixed reference frame $T_2(C,x''_1,y''_1,z''_1)$ and the scalar quantities $J_{x_1''y_1''}, J_{y_1''z_1''}, J_{x_1''z_1''}$ represent the centrifugal moments of inertia relatively to the pairs of plane belonging to the same reference system $T_2(Cx''_1\, y''_1\, z''_1)$.

The angular momentum $\{K_{O_1}\}$ about the centre "O_1" has the following expression:

$$\{K_{O_1}\} = [A]\cdot\{v_1\} \qquad (23)$$

$$[A] = \left[\, [S_{O_1}] \,\vdots\, [J_{O_1}] \,\right] \qquad (24)$$

$$[S_{O_1}] = [M]\cdot[O_1C] \qquad (25)$$

In relation (24) $[S_{O_1}]$ represents the anti-symmetric matrix associated to the vector polar static moment $\{S_{O_1}\}$.The mathematical expression of the vector polar static moment is the following:

$$\{S_{O_1}\} = [M]\cdot\{O_1C\} \qquad (25)$$

$$\{v_1\} = \left[\{v_{O_1}\}^T \,\vdots\, \{\omega_1\}^T \right]^T \qquad (26)$$

$$\{v_{O_1}\} = \left[v_{O_1x_1} \,\vdots\, v_{O_1y_1} \,\vdots\, v_{O_1z_1} \right]^T \qquad (27)$$

$$\{\omega_1\} = \left[\omega_{x_1} \,\vdots\, \omega_{y_1} \,\vdots\, \omega_{z_1} \right]^T \qquad (28)$$

The momentum "H" of the solid rigid body may be written as followings:

$$\{H\} = [M]\cdot\{v_C\} = [B]\cdot\{v_1\} \qquad (29)$$

$$\{v_C\} = \left[v_{Cx_1} \,\vdots\, v_{Cy_1} \,\vdots\, v_{Cz_1} \right]^T \qquad (30)$$

$$[B] = \left[\, [M] \,\vdots\, -[S_{O_1}] \,\right]^T \qquad (31)$$

$$[J_{O_1}] = \begin{bmatrix} J_{x_1} & -J_{x_1y_1} & -J_{x_1z_1} \\ -J_{x_1y_1} & J_{y_1} & -J_{y_1z_1} \\ -J_{x_1z_1} & -J_{y_1z_1} & J_{z_1} \end{bmatrix} \qquad (32)$$

In relationship (32) the scalar quantities $J_{x_1}, J_{y_1}, J_{z_1}$, represent the axial moments of inertia relatively to the axes of the body fixed reference frame $T_1(O_1x_1y_1z_1)$ and the scalar quantities $J_{x_1y_1}, J_{y_1z_1}, J_{x_1z_1}$ represent the centrifugal moments of inertia relatively to the pairs of plane belonging to the same reference system T_1 $(O_1x_1y_1z_1)$.

The relationship (15) may be written in the following form:

$$[A]\cdot\{v_1\} = [D]\cdot\{v_1\} + [C]\cdot\{v_1\} \qquad (33)$$

In relationship (33) matrices [C] and [D] has the following expressions:

$$[C] = \left[[S_{O_1}] \,\vdots\, [M]\cdot[O_1C]\cdot[O_1C]^T \right] \qquad (34)$$

$$[D] = \left[\underbrace{[0]}_{3\times3} \,\vdots\, [J_C] \right] \qquad (35)$$

From the above relationship may be deduced the following relationship between the matrices [A], [C] and [D]:

$$[A] = [D] + [C] \qquad (36)$$

From the relationship (36) it may be deduce the following relationship:

$$[J_{O_1}] = [J_C] + [M][O_1C][O_1C]^T \qquad (37)$$

The matrix relationship (37) represents the mathematical expression of Steiner's theorem.

6. CONCLUSIONS

Relationship (13) is analogous with the relationship which expresses the moments theorem in statics. The relationship which expresses moments theorem in statics is the following:

$$\{M_{O_1}\} = \{M_{O_2}\} + [O_1O_2]\{R\} \qquad (38)$$

In relationship (38) the quantities involved have the following meanings:

$\{M_{O_1}\}$ - represents the resultant moment vector of an arbitrary force system about the arbitrary point "O_1". The point "O_1" is called the old reduction center.

$\{M_{O_2}\}$ - represents the resultant moment vector of an arbitrary force system about the arbitrary point "O_2". The point "O_2" is called the new reduction center.

$\{R\}$ - represents the resultant force vector of an arbitrary system of forces which is an invariant with respect to the choice of the reduction center.

The vector quantities involved in relationship (13) have the following meanings:

$\{K_{O_1}\}$ - represents the resultant angular moment vector of an arbitrary system of angular vectors about the arbitrary point "O_1".

$\{K_{O_2}\}$ - represents the resultant angular moment vector of an arbitrary system of angular vectors about the arbitrary point "O_2".

$\{H\}$ - represents the momentum of the solid rigid body in general motion which is equal to the momentum of its center of mass under the assumption that the total mass of the body is concentrated at that centre

The relationship (13) may be used at the deduction of angular momentum vector about any point which belongs or not to the solid rigid body. For instance we can determine the angular momentum vector about the fixed point "O":

$$\{K_O\} = \{K_{O_1}\} + [OO_1]\{H\} \quad (39)$$

In relationship (38) the expression of the momentum vector $\{H\}$ is given by the relation (29).

The relationship (38) can be used to deduct relationships that describe the general movement of a solid rigid body.

Relation (38) may be written briefly as follows:

$$\{K_O\} = [N] \cdot \{v_1\} \quad (40)$$

The terms involved in relationship (39) have the following expressions:

$$[N] = [N_1] + [N_2] \quad (41)$$

$$[N_1] = \left[[S_{O_1}] \,\vdots\, [J_{O_1}] \right] \quad (42)$$

$$[N_2] = \left[[OO_1] \cdot [M] \,\vdots\, -[OO_1] \cdot [S_{O_1}] \right] \quad (43)$$

The expression of vector $\{v_1\}$ is given by the relationships (26)-(28).

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