NEW ASPECTS ON KOENIG'S THEOREM FOR KINETIC ENERGY

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Abstract.. *The paper presents a method of deduction of the kinetic energy computing relationship for the rigid solid in general motion and from this is deduced the mathematical relation of the Koenig's theorem for kinetic energy. Further on, is presented a demonstration of the Steiner's theorem which is based on the mathematical expression of Koenig's theorem in case of kinetic energy*

Keywords: *kinetic energy, computing relationship, mass center, Koenig's theorem*

1. INTRODUCTION

Let us consider a solid rigid body as is shown in figure 1. This solid rigid body is supposed to be in general motion. Let us choose a rectangular Cartesian system T(O x y z) in space which we conditionally regard as being fixed. The position of a rigid body in space can be specified by the positions of its three points not lying in one straight line [1, 2, 3]. To this end, let us take a coordinate system $T_1(O_1x_1y_1z_1)$ thought of being rigidly connected with the rigid body. The position of the coordinate system $T_1(O_1x_1y_1z_1)$ determines that of the body itself to the fixed reference frame T(O x y z)[6, 7, 8].

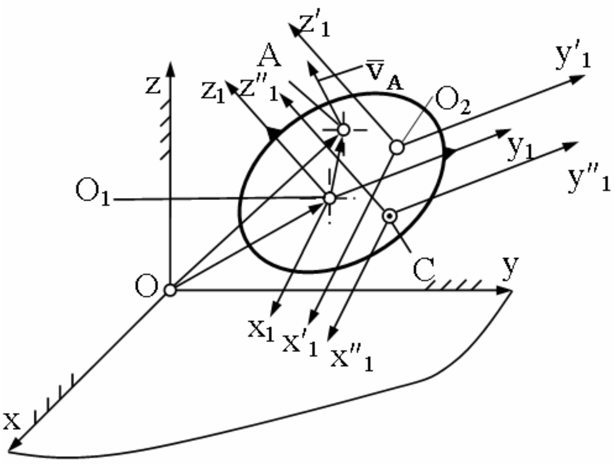

Fig. 1 Solid rigid body in general motion

It will also consider two other reference systems $T_2(C,x''_1,y''_1,z''_1)$ and $T_3(O_2, x'_1, y'_1, z'_1)$ which are also thought as being rigidly connected with the solid rigid body. The origin of the reference frame $T_2(C,x''_1,y''_1,z''_1)$ is a specific point of the body namely its mass center C. Point O_2 is an arbitrary point of the rigid body. Point A is also an arbitrary point which belongs to the rigid body. The axes of the reference systems $T_1(O_1x_1y_1z_1)$, $T_2(C,x''_1,y''_1,z''_1)$ and $T_3(O_2x'_1\ y'_1\ z'_1)$ are considered to be parallel to each other.

We propose to calculate the kinetic energy of the solid rigid body which is considered to be in general motion.

2. KINETIC ENERGY DETERMINATION

The kinetic energy of the solid rigid body may be calculated with the following relation [3, 4, 5, 9, 10-15]:

$$E_C = (1/2)\cdot\int_{(C)}\{v_A\}^T\{v_A\}dm \tag{1}$$

where:

(C) – represents the area occupied by the rigid solid body

$\{v_A\}$ – the velocity vector of the current point "A" of the rigid body

The relationship (1) may be written in the following form:

$$E_C = (1/2)\cdot[L]\cdot\{v_1\} \tag{2}$$

In relation (2) the matrices [L] and $\{v_1\}$ have the following expressions:

$$[L] = \left[\{H\}^T \,\vdots\, \{K_{O_1}\}^T\right] \tag{3}$$

$$\{v_1\} = \left[\{v_{O_1}\}^T \,\vdots\, \{\omega_1\}^T\right]^T \tag{4}$$

In relation (3) {H} represents the momentum vector of the solid rigid body and $\{K_{O_1}\}$ represents the angular momentum vector of the solid rigid body about the centre "O_1". They have the following expressions:

$$\{H\} = [A]\cdot\{v_1\} \tag{5}$$

$$\{K_{O_1}\} = [B]\cdot\{v_1\} \tag{6}$$

$$[A] = \left[[M] \,\vdots\, -[S_{O_1}]\right] \tag{7}$$

$$[B] = \left[[S_{O_1}] \,\vdots\, [J_{O_1}]\right] \tag{8}$$

$$[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \tag{9}$$

$$\left[S_{O_1}\right] = [M]\cdot[O_1C] \tag{10}$$

$$[O_1C]=\begin{bmatrix} 0 & -\zeta_1 & \eta_1 \\ \zeta_1 & 0 & -\xi_1 \\ -\eta_1 & \xi_1 & 0 \end{bmatrix} \tag{11}$$

$$\left[J_{O_1}\right] = \begin{bmatrix} J_{x_1} & -J_{x_1y_1} & -J_{x_1z_1} \\ -J_{x_1y_1} & J_{y_1} & -J_{y_1z_1} \\ -J_{x_1z_1} & -J_{y_1z_1} & J_{z_1} \end{bmatrix} \tag{12}$$

In relation (10) $\left[S_{O_1}\right]$ represents the anti-symmetric matrix associated with the polar static moment vector $\left\{S_{O_1}\right\}$. Its expression is the following:

$$\left\{S_{O_1}\right\} = [M]\cdot\{O_1C\} \tag{13}$$

$$\{O_1C\} = [\xi_1 \mid \eta_1 \mid \zeta_1]^T \tag{14}$$

In relationship (14) the scalar quantities ξ_1, η_1, ζ_1 represent the coordinates of the mass center "C" relatively to the body fixed reference frame $T_1(O_1x_1y_1z_1)$

In relationship (12), the scalar quantities $J_{x_1}, J_{y_1}, J_{z_1}$, represent the axial moments of inertia relatively to the axes of the body fixed reference frame $T_1(O_1,x_1,y_1,z_1)$ and the scalar quantities $J_{x_1y_1}, J_{y_1z_1}, J_{x_1z_1}$ represent the centrifugal moments of inertia relatively to the pairs of plane belonging to the same reference system $T_1(O_1x_1y_1z_1)$.

3. KOENIG'S THEOREME DEDUCTION IN THE CASE OF KINETIC ENERGY

The relationship (1) can be written as followings:

$$E_C = (1/2)\{H\}^T\left\{v_{O_1}\right\} + (1/2)\left\{K_{O_1}\right\}^T\{\omega_1\} \tag{15}$$

In relation (15) the vector quantities $\left\{v_{O_1}\right\}$ and $\{\omega_1\}$ have the following expressions:

$$\left\{v_{O_1}\right\} = \left[v_{O_1x_1} \mid v_{O_1y_1} \mid v_{O_1z_1}\right]^T \tag{16}$$

$$\{\omega_1\} = \left[\omega_{x_1} \mid \omega_{y_1} \mid \omega_{z_1}\right]^T \tag{17}$$

According to Koenig's theorem in the case of angular momentum the following relationship may be written:

$$\left\{K_{O_1}\right\} = \{K_C\} + [O_1C]\{H\} \tag{18}$$

By replacing the relationship (18) in the relation (15) and making calculations we will obtain:

$$E_C = (1/2)\{H\}^T\{v_C\} + (1/2)\{K_C\}^T\{\omega_1\} \tag{19}$$

In the relation (19) the vector quantity $\{v_C\}$ represents the velocity of the mass center of the solid rigid body and the vector quantity $\{K_C\}$ represents the angular momentum of the solid rigid body about its mass center "C". They have the following mathematical expressions:

$$\{v_C\} = \left[v_{Cx_1} \mid v_{Cy_1} \mid v_{Cz_1}\right]^T \tag{20}$$

$$\{K_C\} = [J_C]\cdot\{\omega_1\} \tag{21}$$

In relationship (21) the matrix $[J_C]$ has the following expression:

$$[J_C] = \begin{bmatrix} J_{x_1''} & -J_{x_1''y_1''} & -J_{x_1''z_1''} \\ -J_{x_1''y_1''} & J_{y_1''} & -J_{y_1''z_1''} \\ -J_{x_1''z_1''} & -J_{y_1''z_1''} & J_{z_1''} \end{bmatrix} \tag{22}$$

In the relationship (22) scalar quantities $J_{x_1''}, J_{y_1''}, J_{z_1''}$ represent the axial moments of inertia relatively to the axes of the body fixed reference frame $T_2(C,x''_1,y''_1,z''_1)$ and the scalar quantities $J_{x_1''y_1''}, J_{y_1''z_1''}, J_{x_1''z_1''}$ represent the centrifugal moments of inertia relatively to the pairs of plane belonging to the same reference system $T_2(Cx''_1\ y''_1\ z''_1)$.

Relationship (19) represents the mathematical expression of Koenig's theorem in the case of kinetic energy.

4. DEDUCTION OF STEINER RELATIONSHIP USING KOENIG'S THEOREM IN THE CASE OF KINETIC ENERGY

If we subtract relation (19) from equation (15) we obtain the following relationship:

$$\{v_1\}^T[F]\{v_1\} + \{v_1\}^T[B]^T[G]\{v_1\} = 0 \tag{23}$$

The relationship (23) may be written synthetic in the following way:

$$\{v_1\}^T[N]\{v_1\} = 0 \tag{24}$$

In the relationship (24) the matrix [N] may be expressed as followings:

$$[N] = [F] + [B]^T[G] \tag{25}$$

In the relationship (23) the terms involved has the following expressions:

$$[F] = [C]\cdot[D] \tag{26}$$

$$[C] = \left[[M]^T \mid \left[S_{O_1}\right]^T\right]^T \tag{27}$$

$$[D] = \left[\underbrace{[0]}_{3\times 3} \mid [O_1C]\right] \tag{28}$$

$$[B] = \left[\left[S_{O_1}\right] \mid [A]\right] \tag{30}$$

$$[A] = \left[J_{O_1}\right] - [J_C] \quad (31)$$

$$[G] = \left[\begin{array}{c:c} \underbrace{[0]}_{3\times 3} & [I_3] \end{array}\right] \quad (32)$$

$$[0] = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right] \quad (33)$$

$$[I_3] = \left[\begin{array}{c:c:c} 1 & 0 & 0 \\ \hdashline 0 & 1 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right] \quad (34)$$

In the above relationship the exponent “T” indicates the matrix transposition operation.

From relationship (24) results:

$$[N] = \underbrace{[0]}_{6\times 6} \quad (35)$$

$$\underbrace{[0]}_{6\times 6} = \left[\begin{array}{c:c} \underbrace{[0]}_{3\times 3} & \underbrace{[0]}_{3\times 3} \\ \hdashline \underbrace{[0]}_{3\times 3} & \underbrace{[0]}_{3\times 3} \end{array}\right] \quad (36)$$

Substituting the relationship (25) in the relationship (35) we obtain:

$$[F] + [B]^T [G] = \underbrace{[0]}_{6\times 6} \quad (37)$$

where:

$$[F] = \left[\begin{array}{c:c} \underbrace{[0]}_{3\times 3} & \left[S_{O_1}\right] \\ \hdashline \underbrace{[0]}_{3\times 3} & \left[S_{O_1}\right]\cdot[O_1C] \end{array}\right] \quad (38)$$

$$[B]^T [G] = \left[\begin{array}{c:c} \underbrace{[0]}_{3\times 3} & \left[S_{O_1}\right]^T \\ \hdashline \underbrace{[0]}_{3\times 3} & [A] \end{array}\right] \quad (39)$$

Substituting the relationships (38) and (39) in the relationship (37) we obtain:

$$\left[S_{O_1}\right] + \left[S_{O_1}\right]^T = \underbrace{[0]}_{3\times 3} \quad (40)$$

$$[A] + \left[S_{O_1}\right]\cdot[O_1C] = \underbrace{[0]}_{3\times 3} \quad (41)$$

We note that that equation (40) is actually an identity and therefore it is not important.

Substituting the relationship (31) in the relationship (41) and performing the calculations we obtain:

$$\left[J_{O_1}\right] = [J_C] + [M]\cdot[O_1C]\cdot[O_1C]^T \quad (42)$$

The matrix relationship (42) represents the mathematical expression of Steiner’s theorem.

In the relationship (42) we introduce the following notation:

$$[P] = [O_1C]\cdot[O_1C]^T \quad (43)$$

Using the notation given by the relation (43) the relation (42) can be written as followings:

$$\left[J_{O_1}\right] = [J_C] + [M]\cdot[O_1C]\cdot[O_1C]^T \quad (44)$$

The expression of the matrix [P] is the following:

$$[P] = \left[\begin{array}{c:c:c} \eta_1^2+\zeta_1^2 & -\xi_1\eta_1 & -\xi_1\zeta_1 \\ \hdashline -\xi_1\eta_1 & \xi_1^2+\zeta_1^2 & -\eta_1\zeta_1 \\ \hdashline -\xi_1\zeta_1 & -\eta_1\zeta_1 & \xi_1^2+\eta_1^2 \end{array}\right] \quad (45)$$

5. CONCLUSIONS

The vector quantity $\{K_C\}$ which is given by relation (21) represents the angular momentum of the solid rigid body characterizing its relative motion around its mass center. The relative motion of the solid rigid body around its center of mass may be regarded as a motion of a rigid body about a fixed point.

The relation (19) may be written as followings:

$$E_C = (1/2)\cdot\{v\}^T\cdot[M_C]\cdot\{v\} \quad (46)$$

In relation (46) the quantities involved have the followings expressions:

$$[M_C] = \left[\begin{array}{c:c} [M] & \underbrace{[0]}_{3\times 3} \\ \hdashline \underbrace{[0]}_{3\times 3} & [J_C] \end{array}\right] \quad (47)$$

$$\{v\} = \left[\begin{array}{c:c} \{v_C\}^T & \{\omega_1\}^T \end{array}\right]^T \quad (48)$$

The vector quantities $\{v_C\}$ and $\{\omega_1\}$ are by relations (17) and (20) respectively.

Analyzing the relation (19) we can observe that the kinetic energy of a solid rigid body which is supposed to be in general motion can be calculated as a sum between the kinetic energy of its mass center in which the whole mass of the solid rigid body is supposed to be concentrated and the kinetic energy of the solid rigid body which corresponds to its relative motion about its mass center. (mass center of the solid rigid body).

The kinetic energy of the solid rigid body which is supposed to be in general motion can be calculated using the relation (2).

The relationship (2) may be written in the following equivalent form:

$$E_C = (1/2) \cdot \{v_1\}^T \cdot \left[M_{O_1}\right] \cdot \{v_1\} \qquad (49)$$

In relationship (49) the terms involved have the following expressions:

$$\left[M_{O_1}\right] = \left[\begin{array}{c:c} [M] & -\left[S_{O_1}\right] \\ \hdashline \left[S_{O_1}\right] & \left[J_{O_1}\right] \end{array}\right] \qquad (50)$$

The expressions of the matrices $\{v_1\}$, $[M]$, $\left[S_{O_1}\right]$ and $\left[J_{O_1}\right]$ are given by the relationships (4), (9), (10) and (12) respectively.

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