DYNAMIC ANALYSIS OF ELASTIC MOVEMENTS SUPERPOSED TO THE MOTION OF THE RIGID BODY

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Abstract: *The paper aims to present the dynamic survey of elastic motions which are superposed to rigid movements that occur in a mechanical system consisting of two rigid solids. For this we will write the equations of motion for each rigid solid which is considered to be free and then we will write the differential equations which describe the motion of each rigid solid in the presence of active and constraint forces which are considered to be unknown. Constraint forces will then be eliminated from the differential equations system considering the geometrical constraints which are imposed by the link.*

Keywords: *rigid motion, elastic motion, contact forces, geometric constraints*

1. INTRODUCTION

It is considered the rigid solid system shown in the figure below (fig.1) consisting of two rigid solids. The first one runs o rotational motion under the action of the torque M_{m_1} and the second one executes a rectilinear alternative movement. We propose to study the movements the rigid bodies. The movement of the second rigid solid may be regarded as an elastic motion which is superimposed to the rigid motion (the movement described by the first rigid solid).

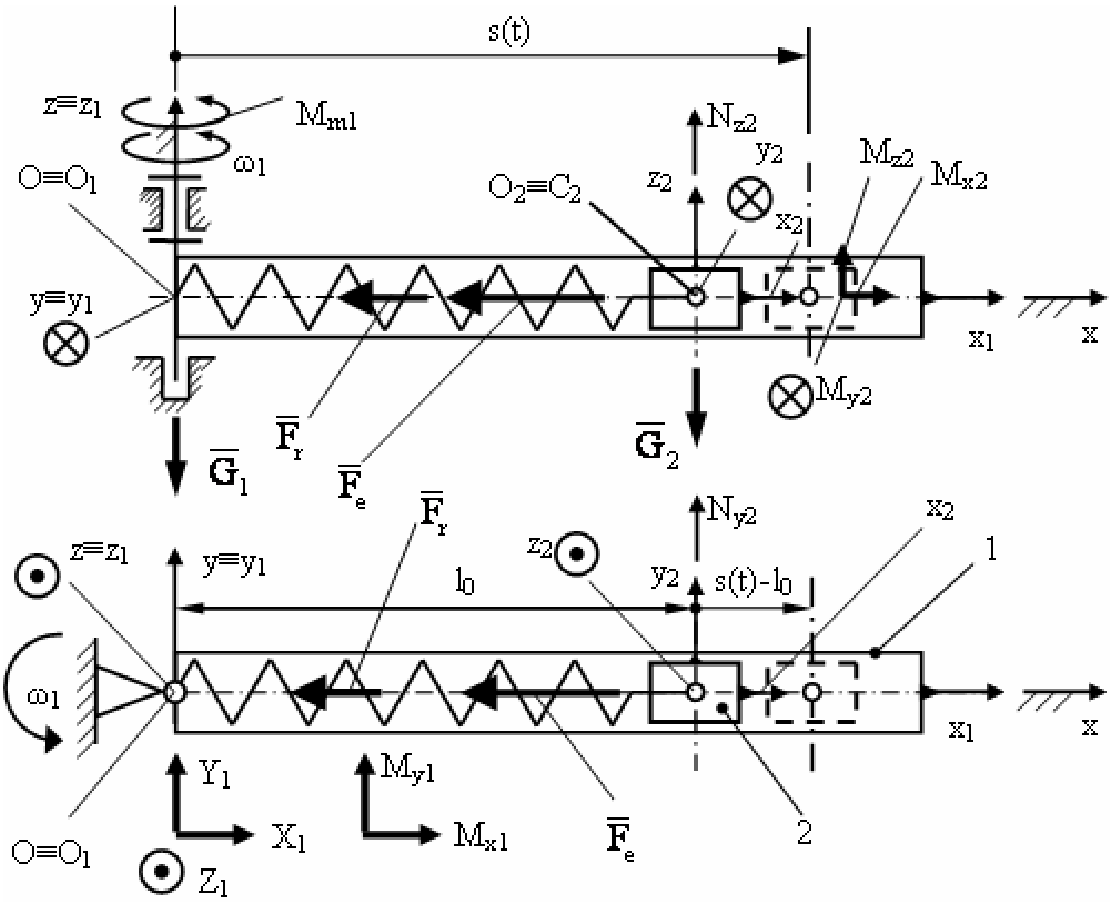

Figure 1. Mechanical system consisting of two rigid solids

2. DETERMINING EQUATIONS OF MOTION FOR THOSE TWO RIGID SOLIDS WHICH ARE CONSIDERED TO BE FREE

For the rigid solid “1”, which is considered to be free, the equations of motion will be written as followings:

$$\left[M_{O_1}\right]\cdot\{\dot{v}_1\} = \left\{Q_1^{g.}\right\} + \{Q_1\} \tag{1}$$

where:

$$\left[M_{O_1}\right] = \left[\begin{array}{c|c} [M_1] & -\left[S_{O_1}\right] \\ \hline \left[S_{O_1}\right] & \left[J_{O_1}\right] \end{array}\right] \tag{2}$$

$$[M_1] = \left[\begin{array}{c|c|c} m_1 & 0 & 0 \\ \hline 0 & m_1 & 0 \\ \hline 0 & 0 & m_1 \end{array}\right] \tag{3}$$

$$\left[S_{O_1}\right] = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \tag{4}$$

$$\{\dot{v}_1\} = \left[\left\{\dot{v}_{O_1}\right\}^T \,\vdots\, \{\dot{\omega}_1\}^T\right]^T \tag{5}$$

$$\left\{\dot{v}_{O_1}\right\} = \left[\dot{v}_{O_1x_1} \,\vdots\, \dot{v}_{O_1y_1} \,\vdots\, \dot{v}_{O_1z_1}\right]^T \tag{6}$$

$$\{\dot{\omega}_1\} = \left[\dot{\omega}_{x_1} \,\vdots\, \dot{\omega}_{y_1} \,\vdots\, \dot{\omega}_{z_1}\right]^T \tag{7}$$

$$\left\{Q_1^{g.}\right\} = -[\Omega_1]\cdot\{v_1\} \tag{8}$$

$\left\{Q_1^{g.}\right\}$- vector of gyroscopic forces

$$[\Omega_1] = \left[\begin{array}{c|c} [\omega_1]\cdot[M_1] & -[\omega_1]\cdot\left[S_{O_1}\right] \\ \hline \left[S_{O_1}\right]\cdot[\omega_1] & [\omega_1]\cdot\left[J_{O_1}\right] \end{array}\right] \tag{9}$$

$$\left[J_{O_1}\right] = \left[\begin{array}{c|c|c} J_{x_1} & 0 & 0 \\ \hline 0 & J_{y_1} & 0 \\ \hline 0 & 0 & J_{z_1} \end{array}\right] \tag{10}$$

$$[\omega_1] = \left[\begin{array}{c|c|c} 0 & -\omega_{z_1} & \omega_{y_1} \\ \hline \omega_{z_1} & 0 & -\omega_{x_1} \\ \hline -\omega_{y_1} & \omega_{x_1} & 0 \end{array}\right] \tag{11}$$

$$\{v_1\} = \left[\left\{v_{O_1}\right\}^T \,\vdots\, \{\omega_1\}^T\right]^T \tag{12}$$

$$\left\{v_{O_1}\right\} = \left[v_{O_1x_1} \,\vdots\, v_{O_1y_1} \,\vdots\, v_{O_1z_1}\right]^T \tag{13}$$

$$\{\omega_1\} = \left[\omega_{x_1} \,\vdots\, \omega_{y_1} \,\vdots\, \omega_{z_1}\right]^T \tag{14}$$

$$\{Q_1\} = \left[\{R_1\}^T \,\vdots\, \left\{M_{O_1}^r\right\}^T\right]^T \tag{15}$$

$$\{R_1\} = [-F_e \,\vdots\, 0 \,\vdots\, -m_1 g]^T + [C]\{\dot{s}\} \tag{16}$$

$$\left\{M_{O_1}^r\right\} = \left[0 \,\vdots\, 0 \,\vdots\, M_{m_1}\right]^T \tag{17}$$

$$M_{m_1} = 10 - \omega_1^2 \tag{18}$$

$$\{\dot{s}\} = [\dot{s} \,\vdots\, 0 \,\vdots\, 0]^T \tag{19}$$

$$[C] = \left[\begin{array}{c|c|c} c & 0 & 0 \\ \hline 0 & c & 0 \\ \hline 0 & 0 & c \end{array}\right] \tag{20}$$

$$\overline{F}_e = -F_e\cdot\overline{i}_1 + 0\cdot\overline{j}_1 + 0\cdot\overline{k}_1 \tag{21}$$

$$F_e = -k\cdot(s - l_0) \tag{22}$$

For the rigid solid “1”, which is considered to be free, the equations of motion will be written as followings:

$$\left[M_{O_2}\right]\cdot\{\dot{v}_2\} = \left\{Q_2^{g.}\right\} + \{Q_2\} \tag{23}$$

In relation (23) the sizes which are involved have the followings expressions:

$$\left[M_{O_2}\right] = \left[\begin{array}{c|c} [M_2] & -\left[S_{O_2}\right] \\ \hline \left[S_{O_2}\right] & \left[J_{O_2}\right] \end{array}\right] \tag{24}$$

$$[M_2] = \left[\begin{array}{c|c|c} m_2 & 0 & 0 \\ \hline 0 & m_2 & 0 \\ \hline 0 & 0 & m_2 \end{array}\right] \tag{25}$$

$$\left[S_{O_2}\right] = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \tag{26}$$

$$\{\dot{v}_2\} = \left[\left\{\dot{v}_{O_2}\right\}^T \,\vdots\, \{\dot{\omega}_2\}^T\right]^T \tag{27}$$

$$\left\{\dot{v}_{O_2}\right\} = \left[\dot{v}_{O_2x_2} \,\vdots\, \dot{v}_{O_2y_2} \,\vdots\, \dot{v}_{O_2z_2}\right]^T \tag{28}$$

$$\{\dot{\omega}_2\} = \left[\dot{\omega}_{x_2} \,\vdots\, \dot{\omega}_{y_2} \,\vdots\, \dot{\omega}_{z_2}\right]^T \tag{29}$$

$$\left\{Q_2^{g.}\right\} = -[\Omega_2]\cdot\{v_2\} \tag{30}$$

$\left\{Q_2^{g.}\right\}$- vector of gyroscopic forces

$$[\Omega_2] = \left[\begin{array}{c|c} [\omega_2]\cdot[M_2] & -[\omega_2]\cdot\left[S_{O_2}\right] \\ \hline \left[S_{O_2}\right]\cdot[\omega_2] & [\omega_2]\cdot\left[J_{O_2}\right] \end{array}\right] \tag{31}$$

$$\left[J_{O_2}\right] = \left[\begin{array}{c|c|c} J_{x_2} & 0 & 0 \\ \hline 0 & J_{y_2} & 0 \\ \hline 0 & 0 & J_{z_2} \end{array}\right] \tag{32}$$

$$[\omega_2] = \left[\begin{array}{c|c|c} 0 & -\omega_{z_2} & \omega_{y_2} \\ \hline \omega_{z_2} & 0 & -\omega_{x_2} \\ \hline -\omega_{y_2} & \omega_{x_2} & 0 \end{array}\right] \tag{33}$$

$$\{v_2\} = \left[\{v_{O_2}\}^T \mid \{\omega_2\}^T\right]^T \quad (34)$$

$$\{v_{O_2}\} = \left[v_{O_2x_2} \mid v_{O_2y_2} \mid v_{O_2z_2}\right]^T \quad (35)$$

$$\{\omega_2\} = \left[\omega_{x_2} \mid \omega_{y_2} \mid \omega_{z_2}\right]^T \quad (36)$$

$$\{Q_2\} = \left[\{R_2\}^T \mid \{M^r_{O_2}\}^T\right]^T \quad (37)$$

$$\{R_2\} = [F_e \mid 0 \mid -m_2 g]^T - [C]\{\dot{s}\} \quad (38)$$

$$\{M^r_{O_2}\} = [0 \mid 0 \mid 0]^T \quad (39)$$

In relation (37), $\{M^r_{O_2}\}$ represents the resultant moment relative to point O_2 of active forces acting on the rigid body “2” from the system

In relation (38) $\{\dot{s}\}$ is given by the following expression:

$$\{\dot{s}\} = [\dot{s} \mid 0 \mid 0]^T \quad (40)$$

3. DETERMINING EQUATIONS OF MOTION FOR THOSE TWO RIGID SOLIDS WHICH ARE CONSIDERED TO BE SUBJECTED TO CONSTRAINTS

For the solid rigid “1” of the system, the equations of motion in the presence of constraints may be written in matrix form as followings:

$$[M_{O_1}] \cdot \{\dot{v}_1\} = \{\tilde{Q}_1\} + \{Q_1\} + \{Q^c_1\} \quad (41)$$

where: $$\{Q^c_1\} = \left[\{R^c_1\} \mid \{M^c_{O_1}\}\right]^T \quad (42)$$

$$\{R^c_1\} = \left[R^c_{11} \mid R^c_{12} \mid R^c_{13}\right]^T \quad (43)$$

$$R^c_{11} = X_1 \quad (44)$$

$$R^c_{12} = Y_1 + N_{y_2} \quad (45)$$

$$R^c_{13} = Z_1 + N_{z_2} \quad (46)$$

$$\{M^c_{O_1}\} = \left[M^c_{11} \mid M^c_{12} \mid M^c_{13}\right]^T \quad (47)$$

$$M^c_{11} = M_{x_1} + M_{x_2} \quad (48)$$

$$M^c_{12} = M_{y_1} + M_{y_2} - s \cdot N_{z_2} \quad (49)$$

$$M^c_{13} = M_{z_2} + s \cdot N_{y_2} \quad (50)$$

In case of the solid rigid “2” of the system, the equations of motion in the presence of constraints may be written in matrix form as followings:

$$[M_{O_2}] \cdot \{\dot{v}_2\} = \{Q^{g.}_2\} + \{Q_2\} + \{Q^c_2\} \quad (51)$$

where: $$\{Q^c_2\} = \left[\{R^c_2\} \mid \{M^c_{O_2}\}\right]^T \quad (52)$$

$$\{R^c_2\} = \left[R^c_{21} \mid R^c_{22} \mid R^c_{23}\right]^T \quad (53)$$

$$R^c_{21} = 0 \quad (54)$$

$$R^c_{22} = -N_{y_2} \quad (55)$$

$$R^c_{23} = -N_{z_2} \quad (56)$$

$$\{M^c_{O_2}\} = \left[M^c_{21} \mid M^c_{22} \mid M^c_{23}\right]^T \quad (57)$$

$$M^c_{21} = -M_{x_2} \quad (58)$$

$$M^c_{22} = -M_{y_2} \quad (59)$$

$$M^c_{23} = -M_{z_2} \quad (60)$$

4. SETTING THE DIFFERENTIAL EQUATIONS THAT DESCRIBE THE MOTION OF THE MECHANICAL SYSTEM

Relations (41) and (52) may be written together as followings:

$$[M] \cdot \{\dot{v}\} = \{Q^{g.}\} + \{Q\} + \{Q^c\} \quad (61)$$

where: $$[M] = \begin{bmatrix} [M_1] & [0] \\ [0] & [M_2] \end{bmatrix} \quad (62)$$

$$\{\dot{v}\} = \left[\{\dot{v}_1\}^T \mid \{\dot{v}_2\}^T\right]^T \quad (63)$$

$$\{Q^{g.}\} = \left[\{Q^{g.}_1\}^T \mid \{Q^{g.}_2\}^T\right]^T \quad (64)$$

$$\{Q\} = \left[\{Q_1\}^T \mid \{Q_2\}^T\right]^T \quad (65)$$

$$\{Q^c\} = \left[\{Q^c_1\}^T \mid \{Q^c_2\}^T\right]^T = [L_\lambda] \cdot \{\lambda\} \quad (66)$$

In relation (61) we perform the followings replacements:

$$\{v\} = [L_\tau]\{\dot{q}\} \quad (67)$$

$$\{\dot{v}\} = [L_\tau]\{\ddot{q}\} + [\dot{L}_\tau]\{\dot{q}\} \quad (68)$$

Substituting relations (67) and (68) in relation (61) and then multiplying the relation (61) to the left by $[L_\tau]^T$ we obtain the following relation:

$$[\tilde{M}] \cdot \{\ddot{q}\} = \{\tilde{Q}^{g.}\} + \{\tilde{Q}\} + \{\tilde{Q}^c\} \quad (69)$$

where: $$[\tilde{M}] = [L_\tau]^T [M][L_\tau] \tag{70}$$

$$\{\tilde{Q}\} = [L_\tau]^T \cdot \{Q\} \tag{71}$$

$$\{\tilde{Q}^c\} = [L_\tau]^T \cdot \{Q^c\} = [0 \mid 0]^T \tag{72}$$

$$\{\tilde{Q}^{g.}\} = -\left([\tilde{\Omega}] + [L_\tau]^T [M][\dot{L}_\tau]\right)\{\dot{q}\} \tag{73}$$

$$\{\ddot{q}\} = [\dot{\omega}_1 \mid \dot{v}_r]^T = [\ddot{\varphi}_1 \mid \ddot{s}]^T \tag{74}$$

$$\{\dot{q}\} = [\omega_1 \mid v_r]^T = [\dot{\varphi}_1 \mid \dot{s}]^T \tag{75}$$

$$[\tilde{\Omega}] = [L_\tau]^T [\Omega][L_\tau] \tag{76}$$

$$[\Omega] = \left[\begin{array}{c|c} [\Omega_1] & [0] \\ \hline [0] & [\Omega_2] \end{array}\right] \tag{77}$$

The matrix $[L_\tau]$ has the following expression:

$$[L_\tau] = [[L_\tau]_1 \mid [L_\tau]_2]^T \tag{78}$$

$$[L_\tau]_1 = \left[\begin{array}{c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{79}$$

$$[L_\tau]_2 = \left[\begin{array}{c|c|c|c|c|c} 0 & s & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{80}$$

$$[\dot{L}_\tau] = [[\dot{L}_\tau]_1 \mid [\dot{L}_\tau]_2]^T \tag{81}$$

$$[\dot{L}_\tau]_1 = \left[\begin{array}{c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{82}$$

$$[\dot{L}_\tau]_2 = \left[\begin{array}{c|c|c|c|c|c} 0 & \dot{s} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{83}$$

Matrix $[L_\tau]$ given by the relation (78) is called in the literature the “orthogonal complement “ of the matrix $[L_\lambda]$. In relation (73) the following notation is introduced:

$$[A] = [\tilde{\Omega}] + [L_\tau]^T [M][\dot{L}_\tau] \tag{84}$$

Using the notation (84) relation (73) becomes:

$$\{\tilde{Q}^{g.}\} = -[A] \cdot \{\dot{q}\} \tag{85}$$

Using the relations (72) si (85) relation (69) becomes:

$$[\tilde{M}] \cdot \{\ddot{q}\} = -[A] \cdot \{\dot{q}\} + \{\tilde{Q}\} \tag{86}$$

In relation (86) the following notation is introduced:

$$\{\dot{q}\} = \{\alpha\} \tag{87}$$

Using the notation given by relation (87), relation (86) becomes:

$$[\tilde{M}] \cdot \{\dot{\alpha}\} = -[A] \cdot \{\alpha\} + \{\tilde{Q}\} \tag{88}$$

Relations (87) si (88) form together a system of four first order differential equations written in matrix form which may be integrated using numerical integration methods and we will obtain the results shown in the figures 2-7. The movement of the mechanical system will be analyzed in two situations: in the presence of structural depreciations and in the presence of structural depreciations.

Thus, in figure 2 is represented the variation with respect to time of the relative displacement in the case of free un-damped oscillations.

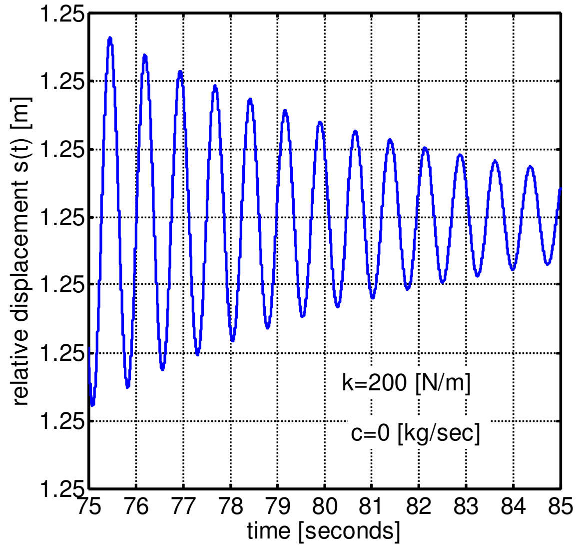

Figure 2. Relative displacement variation with respect to time in the case of un-damped free oscillations

Analyzing the figure it may be observed a periodic variation of the relative motion around a value that corresponds to the situation of relative rest.

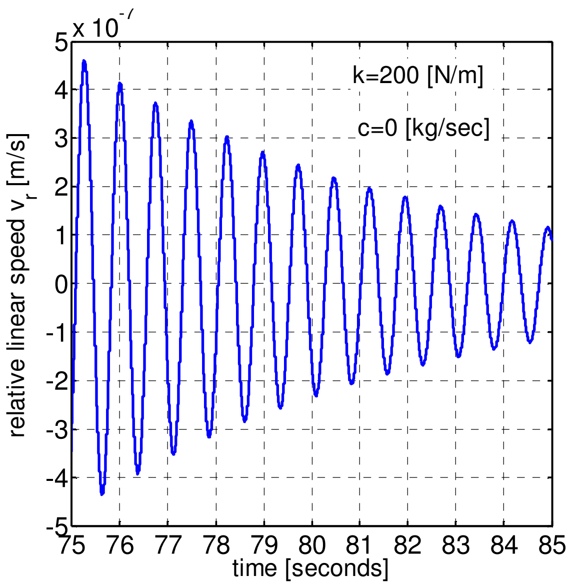

Figure 3. Variation of relative displacement velocity with respect to time in the case of un-damped free oscillations

In the figure 3 is represented the variation of relative displacement speed with respect to time in the case of free un-damped oscillations. Analyzing the figure it may be observed a periodic variation of relative displacement speed around a value that corresponds the situation of relative rest. In the present case this value is zero.

In the figure 4 is represented the angular speed variation with respect to time in the case of free un-damped oscillations of the rigid solid "2". Analyzing the figure it may be observed a periodic variation around a value that corresponds to operation at regime.

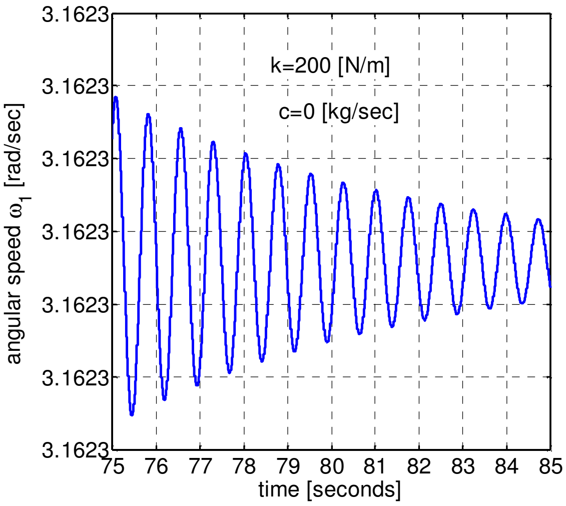

Figure 4. Variation of the angular velocity of rigid body "1" in relation to time in the case of free un-damped oscillations

In the figure 5 is represented the variation of relative displacement with respect to time in the case of free damped oscillations.

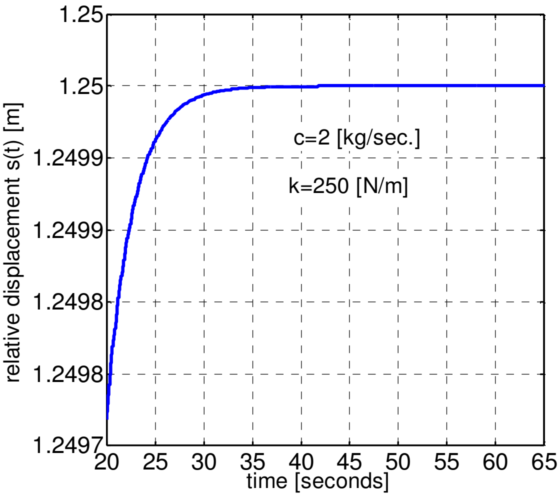

Figure 5. Relative displacement variation with respect to time in the case of damped free oscillations

Analyzing the figure it may be observed that relative linear displacement increases from zero to a value that remains constant, situation corresponding to the relative rest.

In the figure 6 is represented the variation of relative displacement speed with respect to time in the case of free damped oscillations. Analyzing the figure it may be observed that the relative linear displacement velocity decreases to zero value, situation corresponding to the relative rest.

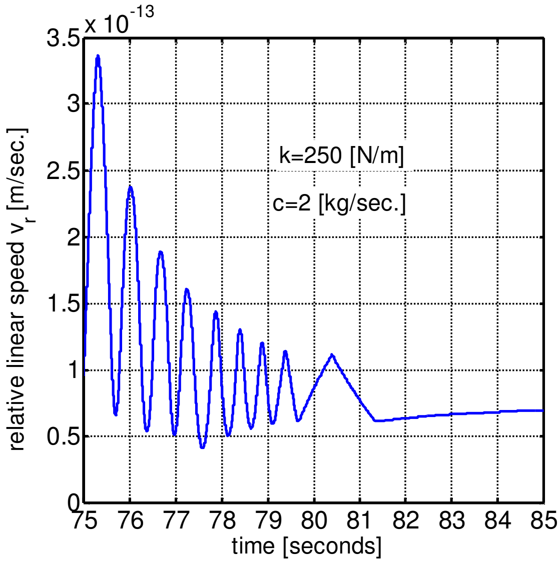

Figure 6. Variation of relative displacement velocity with respect to time in the case of damped free oscillations

In the figure 7 is presented the variation of the angular speed with respect to time of the rigid solid "1" of the system in the case of free damped oscillations. Analyzing the figure it may be observed that the value of angular speed increases from zero to a maximum value that represents the angular speed regime.

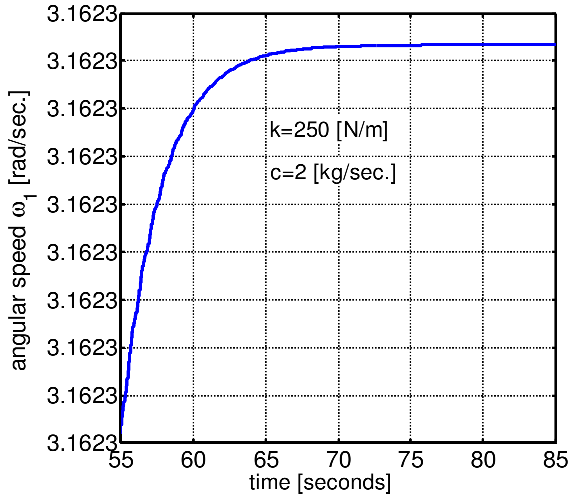

Figure 7. Variation of the angular velocity of rigid body "1" in relation to time in the case of free un-damped oscillations

In conclusion, it may be seen that in the case of free damped oscillations, namely in the presence of structural depreciations, the angular speed of the rigid body "1" of the system stabilizes at a maximum value that is called value of regime and linear velocity characterizing the relative motion of the rigid solid “2” of the system stabilizes at zero value. In other words, the relative motion of the rigid body “2” of the system cancels.

The relative displacement of the rigid solid “2” of the system reaches to a maximum value that corresponds to the situation of relative rest.

5. CONCLUSIONS

The numerical method described in the paper is based on writing in matrix form of differential equations describing the system motion.

Using the numerical method described in the paper one could lead to displacement, velocity and acceleration of any point belonging to those two rigid solids that make up the system.

The system has two interior links: one active inner link which is represented by a linear elastic spring and one inner passive (stationary) which is represented by a slide. The numerical method presented in this paper has a high degree of generality and it can be extended to the dynamic study of any mechanical system met in engineering applications.

6. REFERENCES

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