TORSIONAL VIBRATIONS DURING THE CONSTANT TORQUE START-UP PHASE OF AN INDUSTRIAL EQUIPMENT (I)

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Abstract

The dynamic stresses occurring in the coupling shaft between the motor and the industrial equipment are caused by vibrations due to dynamic loading (inertial shocks or accidental loading). They are superposed on the normal service stresses and can eventually lead to overall loading exceeding the maximum allowable limits. The dynamic stresses are usually considered in the design by means of the overloading coefficient k_s, in order to take this superposition into account. In the case of shaft and gear mechanical systems suddenly loaded with a constant torque, torsional vibrations occur in the shaft causing a dynamic torque M_{td} and shear stresses τ_d which have to be considered in the design. This paper presents a determination method of the dynamic torque and analyses the influence of mass (inertia) and the torsional stiffness paramters of the shaft during the contant torque start-up phase.

Keywords: *constant torque start-up, torsional dynamic stresses*

1. INRODUCTION

An electric motor will be analyzed, having a rotor characterized by the moment of inertia J_1 coupled by means of a shaft to an equipment with the moment of inertia J_2.

The shaft between the motor and the equipment is characterized by the torsional stiffness k_t. The model of the electro-mechanical system is represented in Fig. 1.

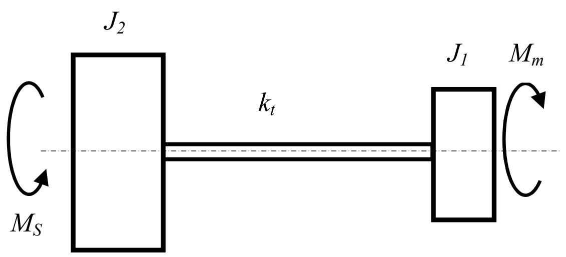

Fig. 1 *The model of the electro-mechanical system*

Obviously, the driving torque M_m required to start-up the equipment has to be larger that the resisting torque M_s:

$$M_m > M_s \tag{1}$$

The start-up phase consists of two important steps (stages):

Step 1: The vibration of the electro-mechanical system, without the motion of the equipment

At the moment $t=0$, the rotor is subjected to a constant driving torque M_m but the equipment is still at rest. The model of this step is presented in Fig. 2.

The following notations are considered in Fig. 2:

M_m –electromagnetic driving torque (constant);

J_1 – moment of inertia of the shaft;

k_t – stiffness parameter of the coupling shaft;

$\varphi_1(t)$ – rotation angle of the rotor.

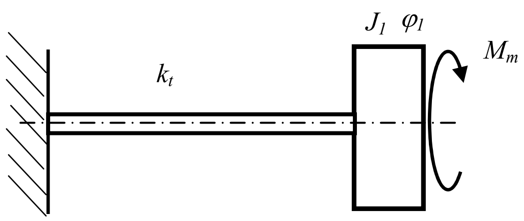

Fig. 2 *The model of the electro-mechanical system for step 1*

Step 1 begins at the moments $t = 0$, when the rotor is subjected to the step torque M_m and the equipment remains for a very short time at rest (Fig. 3).

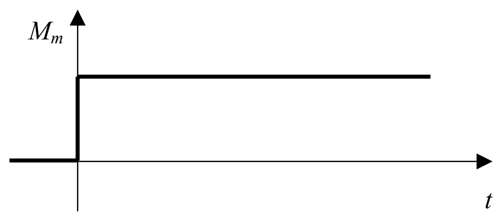

Fig. 3 *The step applied trporque M_m*

The differential equation of the motion of the rotor under the action of the constant driving torque M can be expressed using the theorem of the angular momentum:

$$J_1 \cdot \ddot{\varphi}_1 = -k_t \cdot \varphi_1 + M_m \tag{2}$$

The initial conditions for Step 1 are:

$$\varphi_1(0) = 0;\ \dot{\varphi}_1(0) = 0 \tag{3}$$

Considering the initial conditions (3), the differential equation (2) can be expressed using the angular displacement and the angular velocity:

$$\begin{cases} \varphi_1(t) = \dfrac{M_m}{k_t}\left[1-\cos(p_1 \cdot t)\right] \\ \dot{\varphi}_1(t) = \dfrac{M_m}{k_t} p_1 \sin(p_1 \cdot t) \end{cases} \quad (4)$$

where p_1 is the eigen-frequency of the elastic system:

$$p_1 = \sqrt{\frac{k_t}{J_1}} \quad (5)$$

Note 1: Under the action of the constant torque M_m the shaft will deform under the angle φ_1, producing an internal torque M_s:

$$k_t \cdot \varphi_{1m} = M_s \;\Rightarrow\; \varphi_{1m} = \frac{M_s}{k_t} \quad (6)$$

The time t_1 corresponding to the end of Step 1 is obtained by equalizing the relations (4) and (6):

$$\frac{M_s}{k_t} = \frac{M_m}{k_t}\left(1-\cos\left(t_1\sqrt{\frac{k_t}{J_1}}\right)\right) \quad (7)$$

In this way the end moment of Step 1 t_1 can be determined:

$$t_1 = \sqrt{\frac{J_1}{k_t}} \arccos\left(\frac{M_m - M_s}{M_m}\right) \quad (8)$$

The angular velocity of the rotor at the end of Step 1 is obtained introducing t_1 in relation (4):

$$\dot{\varphi}_1(t_1) = \sqrt{\frac{M_s(2M_m - M_s)}{J_1 \cdot k_t}} \quad (9)$$

The angular displacement given by relation (6) and the velocity of the rotor given by relation (9) are the initial conditions for Step 2 – Setting the equipment into motion.

Note 2: The expression of the angular velocity (9) is valid only if the driving torque $M_m \geq M_s/2$; for the situation when $M_s/2 < M_m < M_s$ system performs an oscillating motion.

Step 2: The vibration of the electro-mechanical system, during the motion of the equipment

The corresponding model of this step is shown in Fig. 4. This is a system with two degrees of freedom, consisting of the rotor with the moment of inertia J_1 subjected to the driving torque M_m, the shaft with the torsional stiffness k_t and the equipment with the equivalent moment of inertia J_2 subjected to the resisting torque M_s $(M_m > M_s)$.

The two rotation angles $\varphi_1(t)$ and $\varphi_2(t)$ are time-independent functions and express the relative motion (the torsional vibrations) of the the rotor and the equipment (of the system with two degrees of freedom) for the situation: $M_s / 2 < M_m < M_s$.

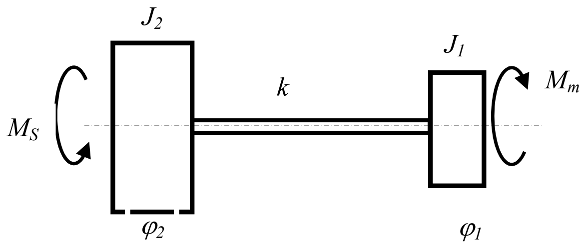

***Fig. 4** The model of the electro-mechanical system for step 2*

The initial conditions for this step (at $t=t_1$) are the final conditions from the Step 1:

- angular displacements:

$$\begin{cases} \varphi_1(t_1) = \dfrac{M_s}{k_t}; \\ \varphi_2(t_1) = 0; \end{cases} \quad (10)$$

- angular velocities:

$$\begin{cases} \dot{\varphi}_1(t_1) = \sqrt{\dfrac{M_s(2M_m - M_s)}{J_1 \cdot k_t}} \\ \dot{\varphi}_2(t_1) = 0 \end{cases} \quad (11)$$

The differential equations of the system motion are obtained using the theorem of the angular momentum:

$$\begin{cases} J_1 \cdot \ddot{\varphi}_1 + k_t(\varphi_1 - \varphi_2) = M_m \\ J_2 \cdot \ddot{\varphi}_2 - k_t(\varphi_1 - \varphi_2) = -M_s \end{cases} \quad (12)$$

We denote by $\varphi = \varphi_1 - \varphi_2$ the relative angle of rotation between the rotor and the equipment.

The differential equation of the relative motion between the rotor and the equipment is obtained by dividing the equations (12) by J_1 and respectively J_2 and subtracting them accordingly:

$$\ddot{\varphi} + k_t \frac{J_1 + J_2}{J_1 \cdot J_2} \cdot \varphi = \frac{J_2 M_m + J_1 M_s}{J_1 \cdot J_2} \quad (13)$$

The general solution of the differential equation (13) is the sum of harmonic homogeneous solution and a particular solution:

$$\begin{cases} \varphi(t) = a \cdot \cos(pt - pt_1) + b \cdot \sin(pt - pt_1) + \\ \qquad + \dfrac{J_2 M_m + J_1 M_s}{(J_1 + J_2) \cdot k_t} \\ \dot{\varphi}(t) = -a \cdot p \cdot \sin(pt - pt_1) + b \cdot p \cdot \cos(pt - pt_1) \end{cases} \quad (14)$$

where p is the eigen-frequency of the relative vibration :

$$p = \sqrt{\frac{J_1 + J_2}{J_1 \cdot J_2} k_t} \quad (15)$$

The constants a and b from the general solution (14) can be determined using the initial conditions (10) and (11) in the differential equation (12):

$$b=\frac{M_s}{k_t}\sqrt{\frac{2M_m-M_s}{M_s}\cdot\frac{J_2}{J_1+J_2}};\qquad a=\frac{M_s}{k_t}\left[1-\frac{J_2M_m+J_1M_s}{M_s\cdot(J_1+J_2)}\right]; \tag{16}$$

The general solution of the differential equation (13) is:

$$\varphi(t)=\frac{M_s}{k_t}\left[\begin{array}{l}\frac{J_2M_m+J_1M_s}{(J_1+J_2)\cdot M_s}+cos(pt-pt_1)+\\+\sqrt{\frac{2M_m-M_s}{M_s}\cdot\frac{J_2}{J_1+J_2}}\cdot sin(pt-pt_1)+\end{array}\right]; \tag{17}$$

$$\dot{\varphi}(t)=\frac{M_s}{k_t}p\cdot\left[\begin{array}{l}\sqrt{\frac{2M_m-M_s}{M_s}\cdot\frac{J_2}{J_1+J_2}}\cdot cos(pt-pt_1)-\\-sin(pt-pt_1)\end{array}\right]$$

The dynamic torsion moment of the coupling shaft is a linear function, depending on the rotation angle $\varphi(t)$:

$$M_d(t)=k_t\cdot\left[c\cdot cos(pt-pt_1-\theta)+d\right] \tag{18}$$

where:

$$c=\frac{M_s}{k_t}\sqrt{1+\frac{2M_m-M_s}{M_s}\cdot\frac{J_2}{J_1+J_2}};\qquad d=\frac{J_2M_m+J_1M_s}{(J_1+J_2)\cdot M_s}; \tag{19}$$

$$\theta=arctg\sqrt{\frac{2M_m-M_s}{M_s}\cdot\frac{J_2}{J_1+J_2}} \tag{20}$$

2. NUMERICAL SIMULATION OF MOTION

Depending on the relative values of driving torque M_m and the resisitng torque M_s, the following particular cases can be anlyzed:

Case 1: $M_m\le\frac{M_s}{2}$ when the equipment is at rest and only the rotor J_1 vibrates, due to the application of the driving torque M_m. Fig. 5 shows the simulated motion using MATHCAD.

The solution of the vibrations is given by equations (4):

$$\varphi_1(t)=\frac{M_m}{k_t}\left[1-cos(p_1\cdot t)\right],\quad p_1=\sqrt{\frac{k_t}{J_1}};\qquad \omega_1(t)=\frac{M_m}{k_t}\cdot p_1\cdot sin(p_1\cdot t) \tag{21}$$

In Fig. 5 the graphs of variation of the angular displacements and velocities are plotted for the following particular parameters: $J_1=10^{-2}$ kgm^2; $J_2=5\cdot10^{-2}$ kgm^2; $k_t=1000Nm$; $M_s=300$ Nm i $M_m=150$ Nm .

Remarks:

Figure 5 shows the variation of the angular deformations of the rotor around an average value equal to the static angular deformation M_m/k_t, with an amplitude of:

$$\Delta\varphi_1=\frac{M_m}{k_t} \tag{22}$$

The angular velocity varies around an average value equal to zero with the amplitude:

$$\Delta\omega_1=\frac{M_m}{k_t}\cdot p_1 \tag{23}$$

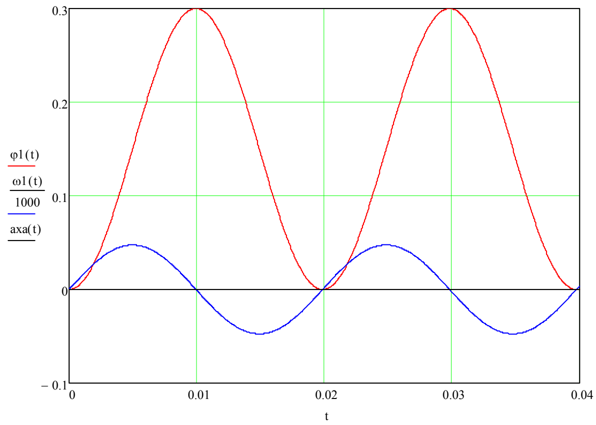

***Fig. 5**. The variation of the angular displacements and velocities for case 1*

Case 2: $\frac{M_s}{2}<M_m<M_s$, when the equipment is not in a solid body motion, but the rotor J_1 and the equipment J_2 vibrate due to the application of driving torque M_m.

At the limit when $M_m=M_s$ the system vibrates; the torsional vibration solution is obtained by replacing $M_m=M_s$ in the relations (17):

$$\varphi(t)=\frac{M_s}{k_t}\left[cos(pt-pt_1)+\sqrt{\frac{J_2}{J_1+J_2}}\cdot sin(pt-pt_1)+1\right] \tag{24}$$

$$\dot{\varphi}(t)=\frac{M_s}{k_t}\cdot p\cdot\left[-sin(pt-pt_1)+\sqrt{\frac{J_2}{J_1+J_2}}\cdot cos(pt-pt_1)\right]$$

Figure 6 shows the variation of the angular displacements and velocities for the following particular parameters: $J_1=10^{-2}$ kgm^2; $J_2=5\cdot10^{-2}$ kgm^2; $k_t=1000Nm$; $M_s=M_m=300$ Nm .

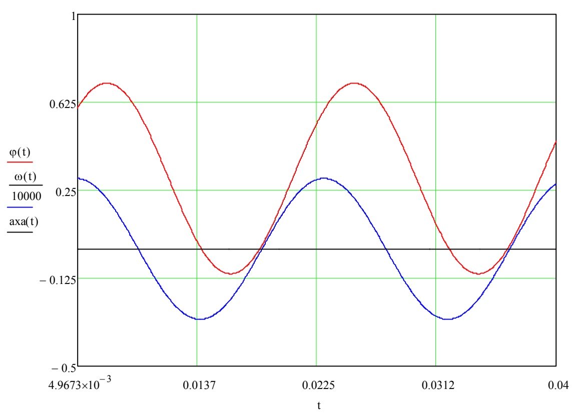

***Fig. 6.** The variation of the angular displacements and velocities for case 2*

Remarks:
The diagrams in Figure 6 show the variation of the angular deformations around an average value equal to the static angular deformation M_m/k_t corresponding to the driving torque, with the amplitude:

$$\Delta\varphi_1 = \frac{M_s}{k_t} = 0{,}3\ rad \qquad (25)$$

The angular velocity varies around an average value equal to zero with the amplitude:

$$\Delta\omega_1 = \frac{M_s}{k_t}\cdot p = 103{,}923\ \ rad/s \qquad (26)$$

The time t_1 given by relation (8) for this case has the value: $t_1 = 4{,}967\cdot 10^{-3}\ s$.

The origin of the time axis in Fig. 6 was considered to be t_1.

Case 3: $M_m > M_s$, when the equipment is in a solid body motion, but the rotor J_1 and the equipment J_2 vibrate due to the application of the driving torque M_m. The solution of the torsional vibrations in this case is given by the relationship (17).

In Fig. 7 the variation of the angular displacements and velocities is plotted using MATHCAD for the following particular parameters:

$J_1=10^{-2}\ kgm^2;\ J_2=5\cdot 10^{-2}\ kgm^2;\ M_m = 1{,}5\cdot M_s = 450Nm$

$k_t=1000Nm;$

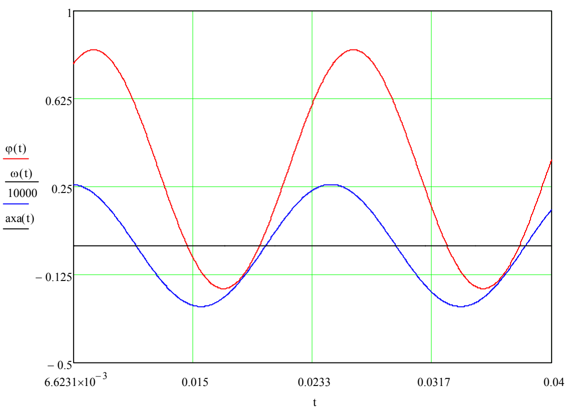

***Fig. 7.** The variation of the angular displacements and velocities for case 3*

Remarks:
The diagrams in Fig. 7 show the variation of the angular deformations around an average value equal to the static angular deformation M_m/k_t corresponding to the driving torque, with the amplitude:

$$\Delta\varphi = \frac{M_s}{k_t} = 0{,}45\ rad \qquad (27)$$

The angular velocity varies around an average value equal to zero, with the amplitude:

$$\Delta\omega = \frac{M_s}{k_t}\cdot p = 155{,}855\ \ rad/s \qquad (28)$$

The time t_1 given by relation (8) for this case has the value: $t_1 = 6{,}623\cdot 10^{-3}\ s$.

The origin of the time axis in Fig. 7 was considered to be t_1.

3. NUMERICAL SIMULATION OF THE DYNAMIC TORQUE IN THE SHAFT

The variation of the dynamic torque in the shaft will be investigated for the three cases mentioned above:

Case 1: $M_m = \frac{M_s}{2} = 150Nm$, when the rotor J_1 and and the equipment J_2 vibrate due to the application of the driving torque M_m producing a deformation $\varphi_1(t)$ given by the relation (21). In this case the dynamic torque in the shaft M_d is given by the following relation:

$$M_d(t) = k_t\cdot\varphi_1(t) = M_m\left[1-\cos(p_1\cdot t)\right]$$
$$p_1 = \sqrt{\frac{k_t}{J_1}} \qquad (29)$$

The variation of the dynamic torque in the shaft M_{d1} is the same as the variation of the angular deformation from Fig. 5, with the difference that the average dynamic torque M_{dmed} and the amplitude of the dynamic torque ΔM_d are equal to the applied driving torque M_m:

$$M_{dmed} = \Delta M_d = 150\ Nm \qquad (30)$$

Case 2: $M_m = M_s = 300Nm$ when the rotor J_1 and the equipment J_2 vibrate due to the application of the driving torque M_m and the deformation of the shaft $\varphi_1(t)$ is given by equation (24). The dynamic torque in the shaft M_d is given by relation:

$$M_d(t) = M_m\left[\cos(pt-pt_1) + \sqrt{\frac{J_2}{J_1+J_2}}\cdot\sin(pt-pt_1)+1\right]; \qquad (31)$$

The average dynamic torque M_{dmed} is equal to the applied driving torque:

$$M_{dmed} = M_m = 300\ Nm\ M_m \qquad (32)$$

The amplitude of dynamic torque ΔM_d is:

$$\Delta M_{d\,max} = M_s\sqrt{\frac{J_1+2J_2}{J_1+J_2}} \qquad (33)$$
$$\Delta M_{d\,max} = 406{,}202\ Nm$$

Case 3: $M_m = 1{,}5\cdot M_s = 450Nm$ when the equipment is in a solid body motion, but the rotor J_1 and the equipment J_2 vibrate due to the application of the driving torque M_m. The deformation of the shaft $\varphi_1(t)$ is given by the relation (17). The dynamic torque M_d is given by the relation:

$$M_d(t) = k_t\cdot\left[c\cdot\cos(pt-pt_1-\theta)+d\right] \qquad (34)$$

where: $c = \frac{M_s}{k_t}\sqrt{1+\frac{2M_m-M_s}{M_s}\frac{J_2}{J_1+J_2}}$

$$d = \frac{J_2 M_m + J_1 M_s}{(J_1+J_2)\cdot M_s}; \qquad (35)$$

The average dynamic torque M_{med} is:

$$M_{dmed} = M_s \cdot \frac{J_2 M_m + J_1 M_s}{(J_1 + J_2) \cdot M_s} \qquad (36)$$

$$M_{dmed} = 325 \ \ Nm$$

The amplitude of dynamic torque ΔM_d is equal with:

$$\Delta M_d = M_s \sqrt{1 + \frac{2M_m - M_s}{M_s} \frac{J_2}{J_1 + J_2}} \qquad (37)$$

$$\Delta M_d = 508{,}675 \ \ Nm$$

4. THE INFLUENCE OF THE MASS AND DRIVING TORQUE PARAMETRERS ON THE DYNAMIC TORQUE IN THE SHAFT

The parameters x and y are definded as it follows:

- The mass parameter x is the ratio of the moments of inertia of the rotor and of the equipment (see Fig. 1):

$$x = \frac{J_1}{J_2} \qquad (38)$$

- The driving torque parameter y is the ratio of the driving torque M_m over the resistant torque M_s (see Fig. 1):

$$y = \frac{M_m}{M_s}, \ y \geq 1 \qquad (39)$$

The maximum and the minimum dynamic torques M_{dmax} and M_{dmin} cand be expressed considering the parameters x and y as it follows:

$$M_{d\,max}(x,y) = M_s\left(\frac{y+x}{1+x} + \sqrt{\frac{2y+x}{1+x}}\right) \qquad (40)$$

$$M_{d\,min}(x,y) = M_s\left(\frac{y+x}{1+x} - \sqrt{\frac{2y+x}{1+x}}\right) \qquad (41)$$

The following quantities will be defined to study the influence of mass parameter x and torque parameter y on the dynamic torque M_d:

- *The dynamic multiplier of the maximum torque:*

$$\Psi_{max} = \frac{M_{d\,max}}{M_s} = \frac{y+x}{1+x} + \sqrt{\frac{2y+x}{1+x}} \qquad (42)$$

- *The dynamic multiplier of the average torque:*

$$\Psi_{med} = \frac{M_{dmed}}{M_s} = \frac{y+x}{1+x} \qquad (43)$$

- *The dynamic multiplier of the torque amplitude*

$$\Delta\Psi = \frac{M_{d\,max} - M_{dmed}}{M_s} = \sqrt{\frac{2y+x}{1+x}} \qquad (44)$$

4.1. The influence of mass parameter *x*

Fig. 8 - 10 show the graphs of variation of all the dynamic multipliers presented above, for the following particular parameters:

$$y = 0.5; \ y = 0.75; \ y = 1; \ y = 2.5; \ y = 4; \quad x \in [0;4] \qquad (45)$$

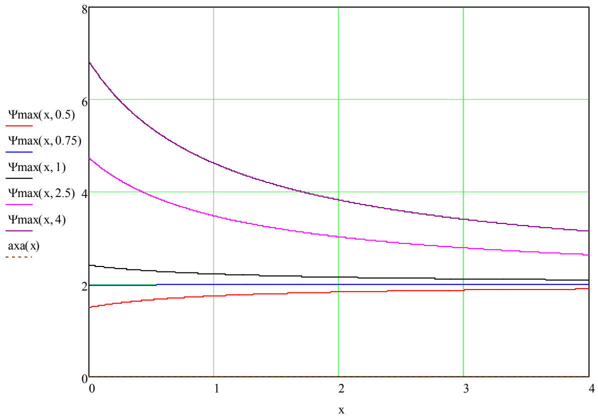

Fig. 8. *Influence of mass parameter x on* Ψ_{max}

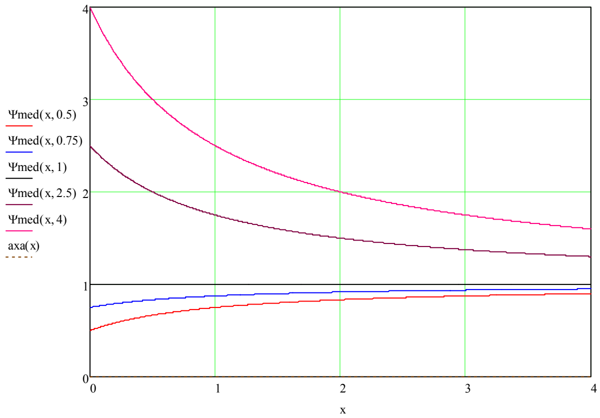

Fig. 9. *Influence of mass parameter x on* Ψ_{med}

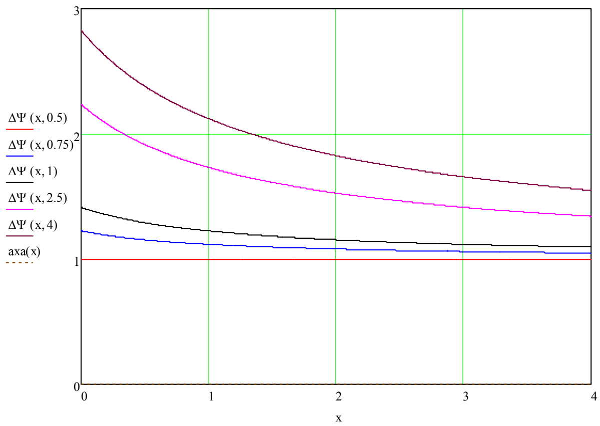

Fig. 10*. Influence of mass parameter x on* $\Delta\Psi$

4.2. The influence of torque parameter *y*

Fig. 11 - 13 show the graphs of variation of the defined multipliers, for the following particular parameters:

$$x = 0.05;\ x = 0.1;\ x = 0.5;\ x = 1;\ \ x = 4; \quad y \in [0;4] \tag{46}$$

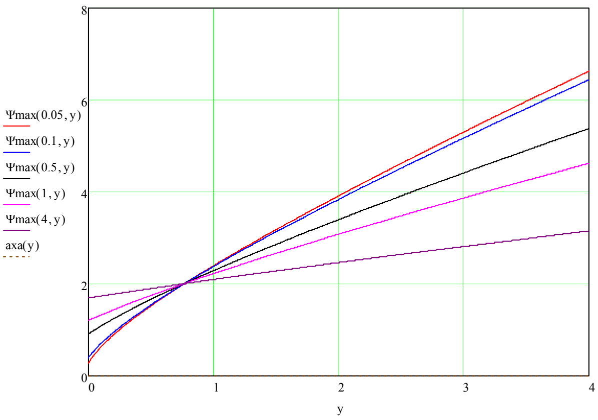

Fig. 11. Influence of torque parameter y on Ψ_{max}

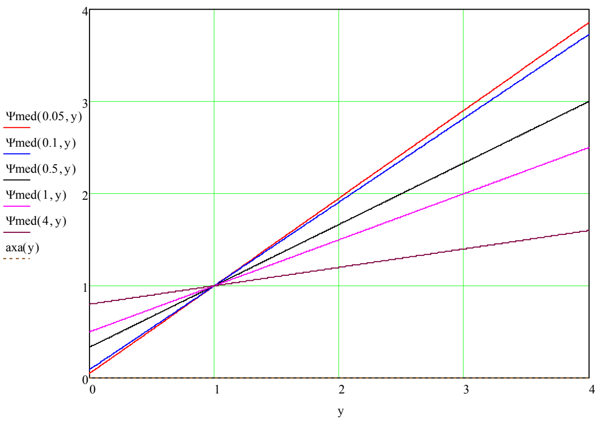

Fig. 12. Influence of torque parameter y on Ψ_{med}

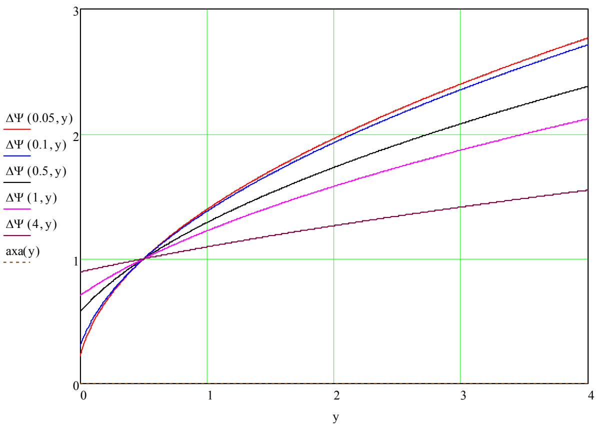

Fig. 13. Influence of torque parameter y on $\Delta\Psi$

5. CONCLUSIONS

The results of the performed theoretical research and numerical simulations, can be summarized in the following conclusions:

- the results of the three insetigated cases show the possibility of setting the system into motion for values of the driving torque between *Ms/2* and M_s, when the driving toruqe M_m is suddenly applied;
- as shown in Fig. 8 - 10, the *dynamic multiplier* $\Delta\Psi$ decreases asymptotically with *x* as well as the *dynamic multiplier* Ψ_{med} for $y \geq 0.5$.
- as shown in Fig. 11 - 13, the *dynamic multipliers* Ψ_{med} and $\Delta\Psi$ increase with the torque parameter *y* and the mass parameter *x*. The lines intersect in a point corresponding to the values: $y = 1$ and respectively $y = 0{,}5$.

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