DYNAMIC STUDY OF A RIGID SOLID WHICH DESCRIBES A PLAN-PARALLEL MOTION SUBJECTED TO LINKS (CONSTRAINTS)

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Abstract: *The paper aims to present a numerical method used to study the dynamic of a rigid solid which describes a plan-parallel motion subjected to links. We will first establish the equations of motion for the rigid solid considered to be free and then we will write the differential equations describing the motion of the rigid solid considering the connection forces that are considerec unknown. The connection forces are then removed from the system of differential equations taking into account the geometrical constraints imposed by the link.*

Keywords: plane-parallel motion dynamic study, connection forces, geometric constraints

1. INTRODUCTION

We consider a rigid solid which describes a plan-parallel motion presented in the figure below (fig.1). A point "O_1" which belongs to the rigid solid is forced to move on a circle of radius "R".

The rigid solid is acted upon by his force of gravity $\overline{G}_1$ which is considered to be an active force. We propose to study the movement of the rigid solid under the action of active forces and the connection forces.

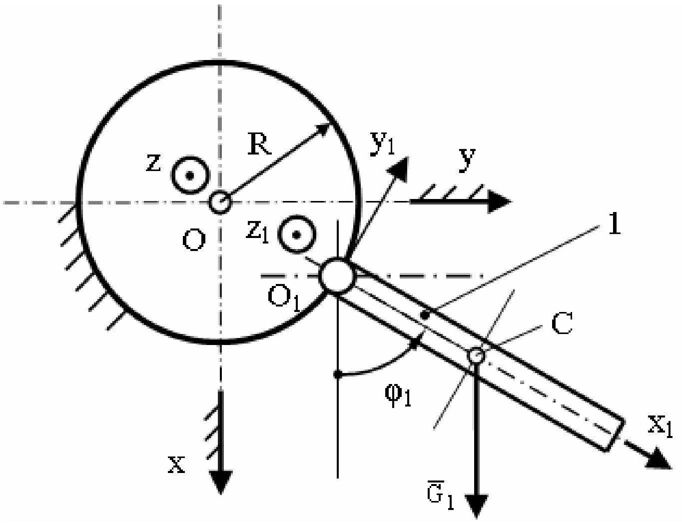

Figure 1. Rigid solid describing a plan-parallel motion

2. ESTABLISHING THE EQUATIONS OF MOTION FOR THE SOLID RIGID BODY CONSIDERED TO BE FREE

For the rigid solid which is supposed to be free, the equations of motion will be written in matrix form as followings:

$$\left[M_{O_1}\right]\cdot\{\dot{v}_1\} = \left\{Q_1^{g.}\right\} + \{Q_1\} \qquad (1)$$

In equation (1) the intervening sizes have the following expressions:

$$\left[M_{O_1}\right] = \left[\begin{array}{c|c} [M_1] & -\left[S_{O_1}\right] \\ \hline \left[S_{O_1}\right] & \left[J_{O_1}\right] \end{array}\right] \qquad (2)$$

$$[M_1] = \left[\begin{array}{c|c|c} m_1 & 0 & 0 \\ \hline 0 & m_1 & 0 \\ \hline 0 & 0 & m_1 \end{array}\right] \qquad (3)$$

$$\left[S_{O_1}\right] = [M_1]\cdot[O_1C] \qquad (4)$$

$$[O_1C] = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & -O_1C \\ \hline 0 & O_1C & 0 \end{array}\right] \qquad (5)$$

$$[J_{O_1}] = \begin{bmatrix} J_{x_1} & 0 & 0 \\ 0 & J_{y_1} & 0 \\ 0 & 0 & J_{z_1} \end{bmatrix} \quad (6)$$

$$\{\dot{v}_1\} = \left[\{\dot{v}_{O_1}\}^T \mid \{\dot{\omega}_1\}^T\right]^T \quad (7)$$

$$\{\dot{v}_{O_1}\} = \left[\dot{v}_{O_1x_1} \mid \dot{v}_{O_1y_1} \mid \dot{v}_{O_1z_1}\right]^T \quad (8)$$

$$\{\dot{\omega}_1\} = \left[\dot{\omega}_{x_1} \mid \dot{\omega}_{y_1} \mid \dot{\omega}_{z_1}\right]^T \quad (9)$$

$$\{Q_1^{g.}\} = -[\Omega_1]\cdot\{v_1\} \quad (10)$$

where:

$$[\Omega_1] = \begin{bmatrix} [\omega_1]\cdot[M_1] & -[\omega_1]\cdot[S_{O_1}] \\ [S_{O_1}]\cdot[\omega_1] & [\omega_1]\cdot[J_{O_1}] \end{bmatrix} \quad (11)$$

$$\{v_1\} = \left[\{v_{O_1}\}^T \mid \{\omega_1\}^T\right]^T \quad (12)$$

$$\{v_{O_1}\} = \left[v_{O_1x_1} \mid v_{O_1y_1} \mid v_{O_1z_1}\right]^T \quad (13)$$

$$\{\omega_1\} = \left[\omega_{x_1} \mid \omega_{y_1} \mid \omega_{z_1}\right]^T \quad (14)$$

$$[\omega_1] = \begin{bmatrix} 0 & -\omega_{z_1} & \omega_{y_1} \\ \omega_{z_1} & 0 & -\omega_{x_1} \\ -\omega_{y_1} & \omega_{x_1} & 0 \end{bmatrix} \quad (15)$$

$$\{Q_1\} = \left[\{R_1\}^T \mid \{M_{O_1}^r\}^T\right]^T \quad (16)$$

$$\{R_1\} = [R_{10}]^T \cdot \{R_1^*\} \quad (17)$$

$$\{R_1^*\} = [0 \mid 0 \mid m_1 \cdot g]^T \quad (18)$$

$$\{M_{O_1}^r\} = [O_1C]\cdot\{R_1\} \quad (19)$$

In equation (1), $\{Q_1^{g.}\}$ is called the vector of “gyroscopic forces” and its expression is given by the relation (10).

3. ESTABLISHING THE EQUATIONS OF MOTION FOR THE SOLID RIGID BODY SUBJECTED TO LINKS

In the presence of links the equations of motion (1) will be written as followings:

$$[M_{O_1}]\cdot\{\dot{v}_1\} = \{Q_1^{g.}\} + \{Q_1\} + \{Q_1^c\} \quad (20)$$

In equation (20), $\{Q_1^c\}$ represents the connection vector forces and it has the following expression:

$$\{Q_1^c\} = [L_\lambda]^T \cdot \{\lambda\} \quad (21)$$

where:

$$\{\lambda\} = [\lambda_1 \mid \lambda_2 \mid \lambda_3 \mid \lambda_4]^T \quad (22)$$

In relation (22), λ_1, λ_2, λ_3, λ_4 represent the Lagrange multipliers which are unknown for now.

4. STABILIREA ECUAȚIILOR DE LEGĂTURĂ ÎNTRE PARAMETRII CINEMATICI

Linking equations between kinematical parameters of the rigid solid “1” may be written under undifferentiated form as followings:

$$x_{O_1}^2 + y_{O_1}^2 + z_{O_1}^2 = R^2 \quad (23)$$

$$z_{O_1} = 0 \quad (24)$$

$$\Phi_x = 0 \quad (25)$$

$$\Phi_y = 0 \quad (26)$$

Under differential form equations (23)-(26) will be written as followings:

$$x_{O_1}\cdot\dot{x}_{O_1} + y_{O_1}\cdot\dot{y}_{O_1} + z_{O_1}\cdot\dot{z}_{O_1} = 0 \quad (27)$$

$$\dot{z}_{O_1} = 0 \quad (28)$$

$$\dot{\Phi}_x = d\Phi_x/dt = \omega_x = 0 \quad (29)$$

$$\dot{\Phi}_y = d\Phi_y/dt = \omega_y = 0 \quad (30)$$

Equations (27)-(30) may be written in matrix form as followings:

$$[L_\lambda^*]\cdot[R_{10}^{ext.}]\cdot\{v_1\} = \{0\} = [0 \mid 0 \mid 0 \mid 0]^T \quad (31)$$

$$[L_\lambda^*] = \begin{bmatrix} [L_\lambda^*]_{11} & [L_\lambda^*]_{12} \\ [L_\lambda^*]_{21} & [L_\lambda^*]_{22} \end{bmatrix} \quad (32)$$

$$[L_\lambda^*]_{11} = \begin{bmatrix} x_{O_1} & y_{O_1} & z_{O_1} \\ 0 & 0 & 1 \end{bmatrix} \quad (33)$$

$$[L_\lambda^*]_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (34)$$

$$[L_\lambda^*]_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (34)$$

$$[L_\lambda^*]_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (35)$$

$$\left[R_{10}^{ext.}\right] = \left[\begin{array}{c|c} [R_{10}] & [0] \\ \hline [0] & [R_{10}] \end{array}\right] \tag{36}$$

$$\left[R_{10}\right] = [\Psi]\cdot[\Theta]\cdot[\Phi] \tag{37}$$

$$[\Psi] = \left[\begin{array}{c|c|c} \cos\Psi & -\sin\Psi & 0 \\ \hline \sin\Psi & \cos\Psi & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \tag{38}$$

$$[\Theta] = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \cos\theta & -\sin\theta \\ \hline 0 & \sin\theta & \cos\theta \end{array}\right] \tag{39}$$

$$[\Phi] = \left[\begin{array}{c|c|c} \cos\varphi & -\sin\varphi & 0 \\ \hline \sin\varphi & \cos\varphi & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \tag{40}$$

$$[0] = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \tag{41}$$

In equation (31) the following notation is introduced:

$$[L_\lambda] = \left[L_\lambda^*\right]\cdot\left[R_{10}^{ext.}\right] \tag{42}$$

Using the notation given by the relation (42) relation (31) will be written as followings:

$$[L_\lambda]\cdot\{v_1\} = \{0\} \tag{43}$$

In relations (38)-(40) the sizes Ψ, θ and φ have the following meanings:

Ψ –angle of precession

θ –angle of nutation

φ –angle of self - rotation

5. ELIMINATION OF LINKING FORCES FROM MOVEMENT EQUATIONS

We multiply the relation (20) to the left with $[L_\tau]^T$ and we will obtain:

$$\left[\tilde{M}_{O_1}\right]\cdot\{\dot{v}_1\} = \left\{\tilde{Q}_1^{g.}\right\} + \left\{\tilde{Q}_1\right\} + \left\{\tilde{Q}_c\right\} \tag{44}$$

where:

$$\left[\tilde{M}_{O_1}\right] = [L_\tau]^T\cdot\left[M_{O_1}\right] \tag{45}$$

$$\left\{\tilde{Q}_1^{g.}\right\} = [L_\tau]^T\cdot\left\{Q_1^{g.}\right\} \tag{46}$$

$$\left\{\tilde{Q}_1\right\} = [L_\tau]^T\cdot\{Q_1\} \tag{47}$$

$$\left\{\tilde{Q}_c\right\} = [L_\tau]^T\cdot\{Q_c\} = \{0\} = [0 \mid 0]^T \tag{48}$$

Taking into account the relation (48), equation (44) becomes:

$$\left[\tilde{M}_{O_1}\right]\cdot\{\dot{v}_1\} = \left\{\tilde{Q}_1^{g.}\right\} + \left\{\tilde{Q}_1\right\} \tag{49}$$

We derive the equation (43) with respect to time and we will obtain:

$$[L_\lambda]\cdot\{\dot{v}\} + \left[\dot{L}_\lambda\right]\cdot\{v\} = \{0\} \tag{50}$$

where:

$$\left[\dot{L}_\lambda\right] = \left[\dot{L}_\lambda^*\right]\cdot\left[R_{10}^{ext.}\right] + \left[L_\lambda^*\right]\cdot\left[\dot{R}_{10}^{ext.}\right] \tag{51}$$

$$\left[\dot{L}_\lambda^*\right] = \left[\begin{array}{c|c} \left[\dot{L}_\lambda^*\right]_{11} & \left[\dot{L}_\lambda^*\right]_{12} \\ \hline \left[\dot{L}_\lambda^*\right]_{21} & \left[\dot{L}_\lambda^*\right]_{22} \end{array}\right] \tag{52}$$

$$\left[\dot{L}_\lambda\right] = \left[\dot{L}_\lambda^*\right]\cdot\left[R_{10}^{ext.}\right] + \left[L_\lambda^*\right]\cdot\left[\dot{R}_{10}^{ext.}\right] \tag{53}$$

$$\left[\dot{L}_\lambda^*\right] = \left[\begin{array}{c|c} \left[\dot{L}_\lambda^*\right]_{11} & \left[\dot{L}_\lambda^*\right]_{12} \\ \hline \left[\dot{L}_\lambda^*\right]_{21} & \left[\dot{L}_\lambda^*\right]_{22} \end{array}\right] \tag{54}$$

$$\left[\dot{L}_\lambda^*\right]_{11} = \left[\begin{array}{c|c|c} \dot{x}_{O_1} & \dot{y}_{O_1} & \dot{z}_{O_1} \\ \hline 0 & 0 & 0 \end{array}\right] \tag{55}$$

$$\left[\dot{L}_\lambda^*\right]_{12} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \tag{56}$$

$$\left[\dot{L}_\lambda^*\right]_{21} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \tag{57}$$

$$\left[\dot{L}_\lambda^*\right]_{22} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \tag{58}$$

$$\left[\dot{R}_{10}^{ext.}\right] = \left[\begin{array}{c|c} [\dot{R}_{10}] & [0] \\ \hline [0] & [\dot{R}_{10}] \end{array}\right] \tag{59}$$

$$\left[\dot{R}_{10}\right] = [R_{10}]\cdot[\omega_1] \tag{60}$$

$$\left[\dot{\Psi}_1 \mid \dot{\theta}_1 \mid \dot{\varphi}_1\right]^T = \left[0 \mid 0 \mid \omega_{z_1}\right]^T \tag{61}$$

$$\left\{\dot{r}_{O_1}\right\} = [R_{10}]\cdot\left\{v_{O_1}\right\} \tag{62}$$

where:

$$\left\{\dot{r}_{O_1}\right\} = \left[\dot{x}_{O_1} \mid \dot{y}_{O_1} \mid \dot{z}_{O_1}\right]^T \tag{63}$$

In relations (45)-(48) the matrix $[L_\tau]$ has the following expression:

$$[L_\tau] = \left[R_{10}^{ext.}\right]^T\cdot\left[L_\tau^*\right] \tag{64}$$

In relation (64) the matrix $\left[L_\tau^*\right]$ has the following expression:

$$\left[L_\tau^*\right] = \left[\left[L_\tau^*\right]_1^T \mid \left[L_\tau^*\right]_2^T\right]^T \tag{65}$$

In relation (65) the matrices $\left[L_\tau^*\right]_1$ and $\left[L_\tau^*\right]_2$ have the followings expressions:

$$\left[L_{\tau}^{*}\right]_{1} = \left[\begin{array}{c|c|c} -y_{O_1} & x_{O_1} & 0 \\ \hline 0 & 0 & 0 \end{array}\right]^{T} \tag{66}$$

$$\left[L_{\tau}^{*}\right]_{1} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]^{T} \tag{67}$$

The matrix $[L_{\tau}]$ defined by the equation (64) is called literature "orthogonal complement". The relations (49), (50), (61) şi (62) form a system of twelve first-order differential equations which are solved using numerical integration methods and we obtain the results shown in the figures below. (fig.2-10).

In figure 2 is shown the variation of the angle size φ_1 with respect to time.

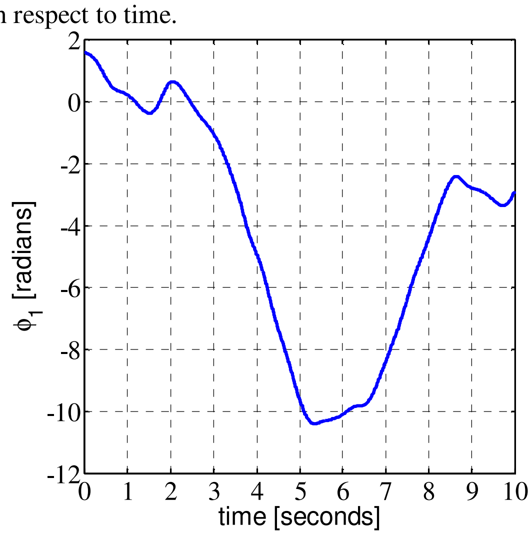

Figure 2.Variation of the angle φ_1 with respect to time

In the figure 3 is shown the variation of angular velocity with respect to time.

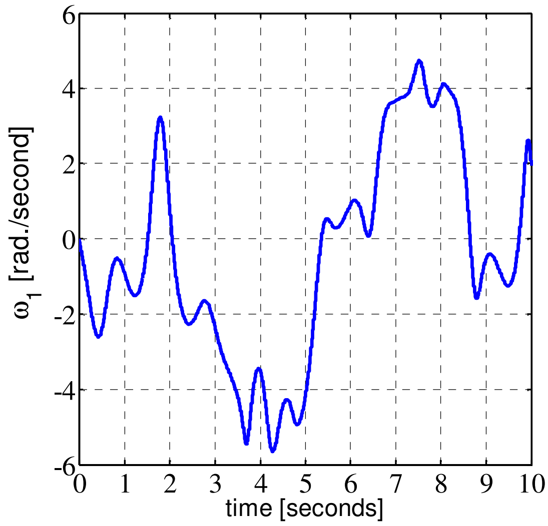

Figure 3.Variation of angular speed ω_1 with respect to time

The rigid solid starts from rest so its initial angular velocity is zero. The dynamic study is performed for a period of ten seconds.
In figure 4 is shown the time evolution of the mobile reference system origin abscissa relatively to the fixed reference frame T(O x y z). The dynamic study is performed for a period of ten seconds. At the initial moment the point "O_1" abscissa (the mobile reference system origin) is zero.

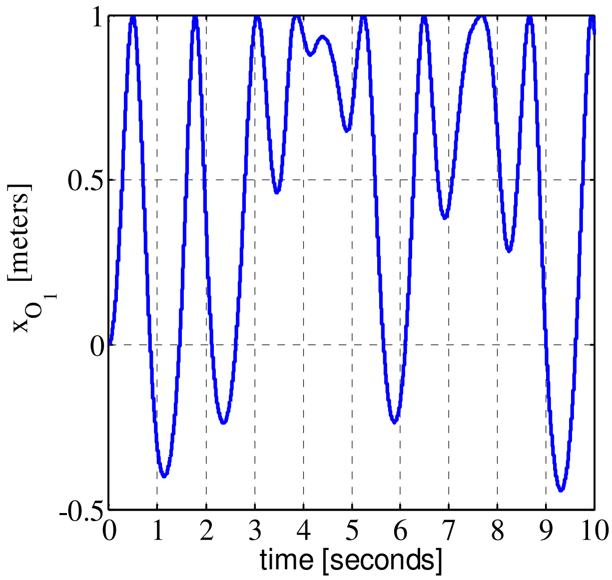

Figure 4. Size variation abscissa origin of the mobile reference system with respect to time

In figure 5 is shown the time evolution of the mobile reference system origin ordinate $T_1(O_1x_1y_1z_1)$ relatively to the fixed reference frame T(O x y z). The dynamic study is carried out over a period of ten seconds. At the initial moment the ordinate of the point "O_1" (the mobile reference system origin) has a value equal to the length of the radius "R" that in the particular case considered has the value R=1.

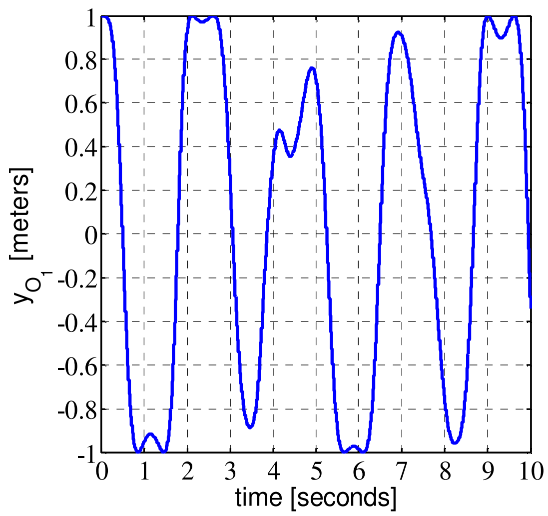

Figure 5. Size variation ordinate origin of the mobile reference system with respect to time

In figure 6 is shown the time evolution of the abscissa value of the rigid body “1” center of mass relatively to the fixed reference system T (O x y z).

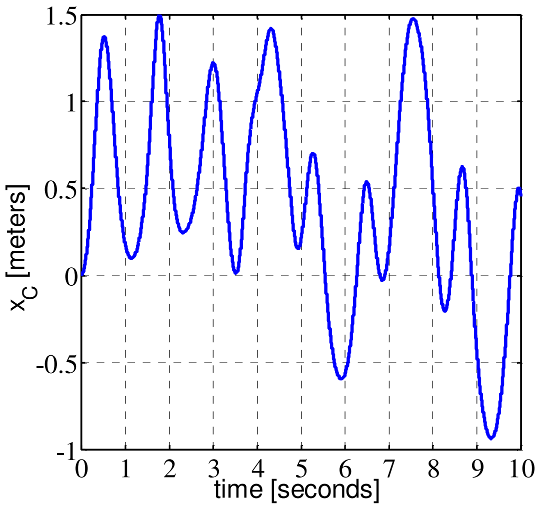

Figure 6. Size variation abscissa of the rigid solid center of mass with respect to time

The dynamic study is carried out over a period of ten seconds. At the initial moment the abscissa of the point “C” (the rigid solid center of mass) is zero.

In figure 7 is shown the time evolution of the ordinate value of the rigid body “1” center of mass relatively to the fixed reference system T (O x y z). The dynamic study is carried out over a period of ten seconds. At the initial moment the ordinate of point “C” (the rigid body center of mass) has a value which is equal to the sum of circle radius and the distance O_1C.

At the initial moment the ordinate value of point “C” (the mass center of the rigid body “1”) may be calculated using the relation:

$$y_{C,0} = R + O_1C \tag{64}$$

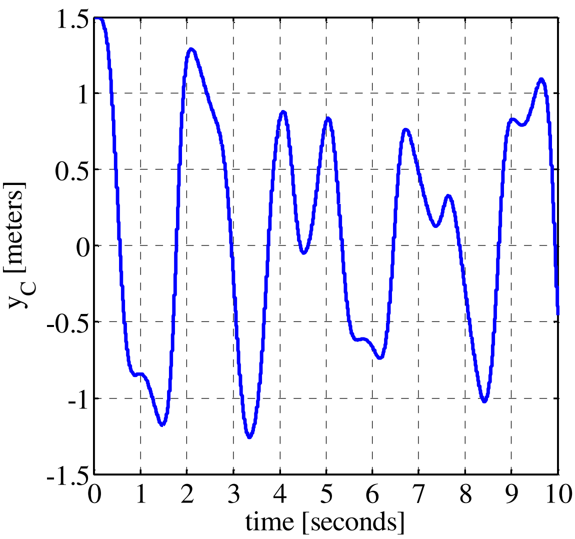

Figure 7. Size variation ordinate of the rigid solid center of mass with respect to time

The rigid body center of mass trajectory is shown in the figure 8.

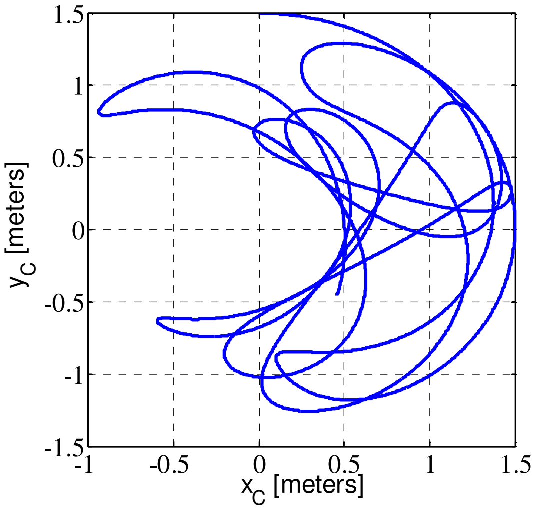

Figure 8. The rigid solid body center of mass trajectory

Analyzing the figure it can be seen that the initial position of the center of mass “C” is characterized by the following coordinates:

$$x_{C,0} = 0, \; y_{C,0} = 1{,}5 \tag{65}$$

The variation of the mobile reference system origin “O1” velocity projections on the mobile reference system $T_1(O_1x_1y_1z_1)$ axes is shown in the figures 9-10. Thus in figure 9 it may be observed the variation with respect to time of the origin “O_1” velocity axis projection O_1x_1. The dynamic study is carried out over a period of ten seconds.

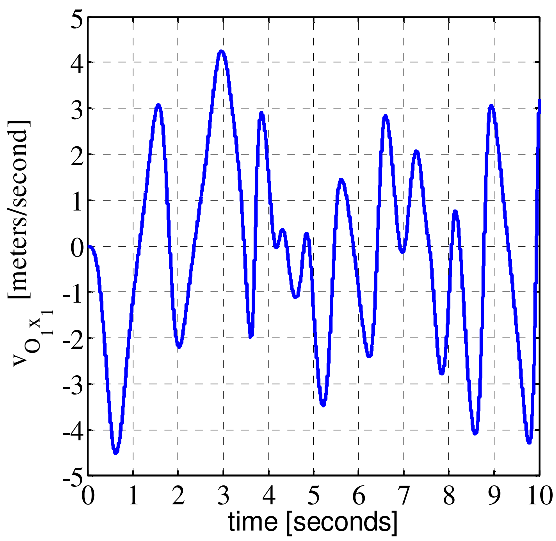

Figure 9. Variation with respect to time of point O_1 velocity axis O_1x_1 projection

In figure 10 it may be observed the variation with respect to time of the origin "O_1" velocity axis projection O_1y_1. The dynamic study is carried out over a period of ten seconds..

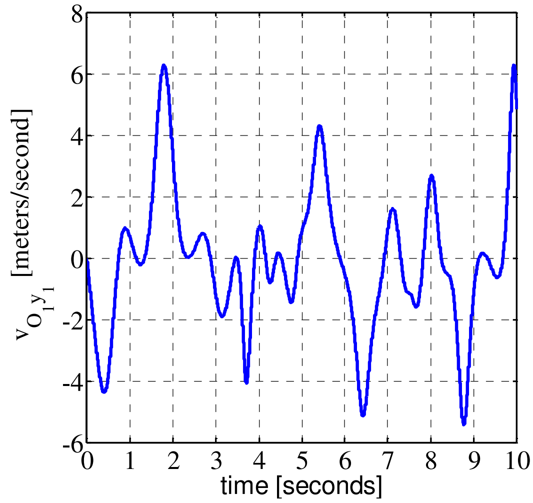

Figure 10. Variation with respect to time of point O_1 velocity axis O_1y_1 projection

6. CONCLUSIONS

The numerical method proposed in the paper is based on writing in matrix form of the differential equations describing the motion rigid solid subjected to links both under the action of active forces (which are considered to be known) and the action of the connecting forces which are considered to be unknown and therefore must be eliminated.

One could determine the displacement velocity and acceleration of any point belonging to the rigid body by using the numerical method described in the paper.

The numerical method presented in this paper has a high degree of generality and it can be extended to the dynamic study of any mechanical system met in engineering applications.

7. REFERENCES

[1] Kossi Atchonouglo, Claude Valée,… Matrix formulation of the motion equations of a rigid body, In: Research Trends in Mechanics volume 3, Editura Academiei Romane, Bucuresti 2009, pp.9 – 29

[2] Mihail Sandu, Mecanica Teoretica, Editura Didactica si Pedagogica Bucuresti, 2002

[3] Valentin Harabor, Mecanica clasica (partea I), Reprografia Universitatii din Craiova, 1999

[4] Iosif Taposu, Mecanica Analitica si Vibratii, Editura Tehnica, Bucuresti, 1998

[5] Stefan Staicu, Mecanica Teoretica, Editura Tehnica, Bucuresti, 1998

[6] L.D. Landau, E.M. Lifsit, Mecanica , Editura Tehnica, Bucuresti 1996

[7] W. Blajer, An orthonormal tangent space method for constrained multibody systems Computer Methods in Applied Mechanics and Engineering 121 (1995) 45-57

[8] W. Blajer, D. Bestle, W. Schiehlen, An Orthogonal Complement Matrix Formulation for Constrained Multibody Systems, Journal of Mechanical Design June 1994, vol.116, pp. 423-428

[9] W. Blajer, A Projection Method Approach to Constrained Dynamic Analysis, Journal of Applied Mechanics 59(1992) 643-649

[10] V.M. Starzhinskii, An Advanced Course of Theoretical Mechanics , Mir Publishers Moscow 1982

[11] Virgil Olariu, Petre Sima, Valeriu Achiriloaie, Mecanica Tehnica, Editura Tehnica Bucuresti 1982

[12] M. Radoi, E. Deciu, Mecanica, Editura Didactica si Pedagogica Bucuresti 1981

[13] Caius Iacob, Mecanica Teoretica, Editura Didactica si Pedagogica Bucuresti, 1980

[14] Targ S., Theoretical Mechanics (A Short Course), Mir Publishers Moscow, 1980

[15] Strelkov S.P., Mechanics, Mir Publishers Moscow, 1978 (in English)

[16] Voinea R. Voiculescu D. Ceausu V. Mecanica, Editura Didactica si Pedagogica Bucuresti 1975

[17] V. Valcovici, St. Balan, R. Voinea, Mecanica Teoretica, Editura Tehnica 1968

[18] Voinaroski R., Mecanica teoretica, Editura Didactica si Pedagogica Bucuresti 1968

[19] Nekrasov A.I., Curs de mecanica teoretica, vol.I-II, Editura Tehnica Bucuresti 1955