A NUMERICAL METHOD USED TO ANALYZE THE DYNAMICS OF A CYCLIC MECHANISM

Vladimir Dragoş Tătaru

University "Valahia" of Târgovişte, Regele Carol I Avenue No.2, Targoviste, Romania

Abstract. *The paper presents a numerical method used to analyze the dynamics of a mechanical system that presents three inner links namely: two elastic links by linear springs and one link by rigid rod of negligible mass. The whole system has three degrees of freedom.*

Keywords: numerical method, mechanical system, cyclic mechanism, rigid rod

1. INTRODUCTION

We will consider the mechanical system which is shown in the figure below (fig.1). The mechanical system consists of four elements whose moments of inertia are J_1, J_2, J_3 and J_4. The first element of the system is acted upon by an electric engine which develops the motor torque M_m. fourth element of the system is acted upon by the resistant torque M_r.

2. DIFFERENTIAL EQUATIONS ESTABLISHMENT FOR EACH ELEMENT OF THE SYSTEM WHICH IS CONSIDERED TO BE FREE

The differential equation that describes the movement of the first element of the system may be written as followings:

$$J_1 \cdot \dot{\omega}_1 = M_m - k_{12} \cdot \Delta\varphi_{12} \qquad (1)$$

where:

$$\Delta\varphi_{12} = \varphi_2 - \varphi_1 \qquad (2)$$

In relation (1) "k_{12}" represents the linear elastic characteristic of the spring.

The differential equation that describes the movement of the second element of the system may be written as followings:

$$J_2 \cdot \dot{\omega}_2 = k_{12} \cdot \Delta\varphi_{12} \qquad (3)$$

The differential equation that describes the movement of the third element of the system may be written as followings:

$$J_3 \cdot \dot{\omega}_3 = -k_{34}\Delta\varphi_{34} \qquad (4)$$

where:

$$\Delta\varphi_{34} = \varphi_3 - \varphi_4 \qquad (5)$$

In relation (4) "k_{34}" represents the linear elastic characteristic of the spring.

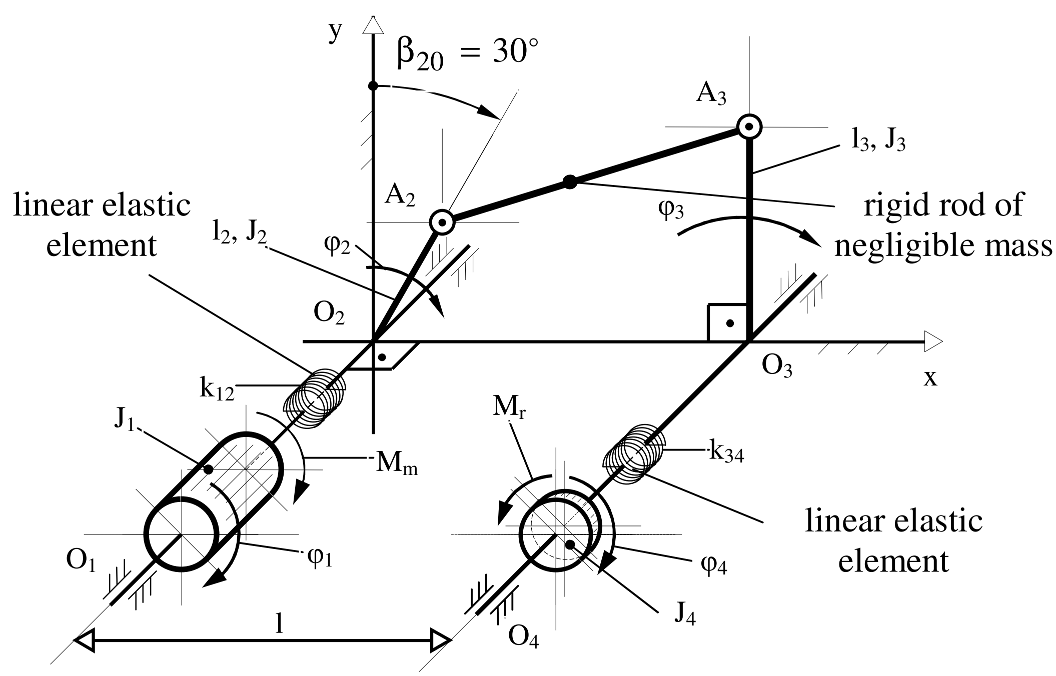

Figure 1 Cyclic mechanism

The differential equation that describes the movement of the fourth element of the system may be written as followings:

$$J_4 \cdot \dot{\omega}_4 = k_{34}\Delta\varphi_{34} - M_r \tag{6}$$

3. ESTABLISHMENT OF THE RELATIONSHIPS BETWEEN KINEMATICAL PARAMETERS OF THE SOLID RIGIDS THAT COMPOUND THE SYSTEM

The relationship between kinematical parameters of the rigid solids "2" and "3" may be written as followings:

$$A_2A_3 = \text{const.} \tag{7}$$

or under the equivalent form:

$$\sqrt{(A_2A_3)_x^2 + (A_2A_3)_y^2} = \text{const.} \tag{8}$$

where:

$$(A_2A_3)_x = x_{A_3} - x_{A_2} \tag{9}$$

$$(A_2A_3)_y = y_{A_3} - y_{A_2} \tag{10}$$

Under differential form the relationship (8) may be written as followings:

$$(A_2A_3)_x\,(A_2\overset{\circ}{A}_3)_x + (A_2A_3)_y\,(A_2\overset{\circ}{A}_3)_y = 0 \tag{11}$$

where:

$$(A_2\overset{\circ}{A}_3)_x = \dot{x}_{A_3} - \dot{x}_{A_2} \tag{12}$$

$$(A_2\overset{\circ}{A}_3)_y = \dot{y}_{A_3} - \dot{y}_{A_2} \tag{13}$$

$$\{A\} = \left[(A_2A_3)_x \;\vdots\; (A_2A_3)_y\right]^T \tag{14}$$

Relationship (11) may be written under the following form:

$$\{A\}^T \cdot \{v\} = 0 \tag{15}$$

where:

$$\{v\} = \{v_{A_2}\} - \{v_{A_3}\} \tag{16}$$

$$\{v_{A_2}\} = [R_2][l_2]\{\omega_2\} \tag{17}$$

$$\{v_{A_3}\} = [R_3][l_3]\{\omega_3\} \tag{18}$$

$$[R_2] = \begin{bmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{bmatrix} \tag{19}$$

$$[l_2] = \begin{bmatrix} 0 & 0 \\ 0 & l_2 \end{bmatrix} \tag{20}$$

$$\{\omega_2\} = [0 \;\vdots\; \omega_2]^T \tag{21}$$

$$[R_3] = \begin{bmatrix} \cos\varphi_3 & -\sin\varphi_3 \\ \sin\varphi_3 & \cos\varphi_3 \end{bmatrix} \tag{22}$$

$$[l_3] = \begin{bmatrix} 0 & 0 \\ 0 & l_3 \end{bmatrix} \tag{23}$$

$$\{\omega_3\} = [0 \;\vdots\; \omega_3]^T \tag{24}$$

$$\{A\}^T = \{r_{A_2}\}^T - \{r_{A_3}\}^T \tag{25}$$

$$\{r_{A_2}\}^T = [x_{A_2} \;\vdots\; y_{A_2}] \tag{26}$$

$$\{r_{A_2}\} = [x_{A_2} \;\vdots\; y_{A_2}]^T \tag{27}$$

$$\{r_{A_3}\}^T = [x_{A_3} \;\vdots\; y_{A_3}] \tag{28}$$

$$\{r_{A_3}\} = [x_{A_3} \;\vdots\; y_{A_3}]^T \tag{29}$$

If we derive relation (25) with respect to time we will obtain:

$$\{\dot{A}\} = \{\dot{r}_{A_2}\}^T - \{\dot{r}_{A_3}\}^T \tag{30}$$

where:

$$\{\dot{r}_{A_2}\} = d\{r_{A_2}\}/dt \tag{31}$$

$$\{\dot{r}_{A_3}\} = d\{r_{A_3}\}/dt \tag{32}$$

$$\{r_{A_2}\} = [R_2]\{l_2\} \tag{33}$$

$$\{r_{A_3}\} = \{O_2O_3\} + [R_3]\{l_3\} \tag{34}$$

$$\{O_2O_3\} = [l \;\vdots\; 0]^T \tag{35}$$

$$\{l_2\} = [l_2 \;\vdots\; 0]^T \tag{36}$$

$$\{l_3\} = [l_3 \;\vdots\; 0]^T \tag{37}$$

$$\{\dot{r}_{A_2}\} = [R_2][\omega_2]\{l_2\} \tag{38}$$

$$\{\dot{r}_{A_3}\} = [R_3][\omega_3]\{l_3\} \tag{39}$$

$$[\omega_2] = \begin{bmatrix} 0 & -\omega_2 \\ \omega_2 & 0 \end{bmatrix} \tag{40}$$

$$[\omega_3] = \begin{bmatrix} 0 & -\omega_3 \\ \omega_3 & 0 \end{bmatrix} \tag{41}$$

Relation (15) may be written as followings:

$$\left(\{r_{A_2}\}^T - \{r_{A_3}\}^T\right) \cdot \{v\} = 0 \tag{42}$$

or :

$$\{B\}^T \cdot \{v_{A_2}\} - \{C\}^T \cdot \{v_{A_3}\} = 0 \tag{43}$$

where:

$$\{B\} = \left(\{l_2\}^T [R_2]^T - \{O_2O_3\}^T - \{l_3\}^T [R_3]^T\right)^T \quad (44)$$

$$\{C\} = \left(\{l_2\}^T [R_2]^T - \{O_2O_3\}^T - \{l_3\}^T [R_3]^T\right)^T \quad (45)$$

Relation (43) may be written under the following form:

$$B_1 \cdot \omega_2 - C_1 \cdot \omega_3 = 0 \quad (46)$$

where:

$$B_1 = \{B\}^T [R_2][l_2][0 \mid 1]^T \quad (47)$$

$$C_1 = \{C\}^T [R_3][l_3][0 \mid 1]^T \quad (48)$$

If we derive relation (46) with respect to time we will obtain:

$$\{D\}^T \cdot \{\dot{\omega}\} + \{\dot{D}\}^T \{\omega\} = 0 \quad (49)$$

where:

$$\{D\} = [B_1 \mid -C_1]^T \quad (50)$$

$$\{\dot{D}\} = [\dot{B}_1 \mid -\dot{C}_1]^T \quad (51)$$

$$\{\dot{\omega}\} = [\dot{\omega}_2 \mid \dot{\omega}_3]^T \quad (52)$$

$$\{\omega\} = [\omega_2 \mid \omega_3]^T \quad (53)$$

$$\begin{aligned}\dot{B}_1 = \{\dot{B}\}^T [R_2][l_2][0 \mid 1]^T + ...\\ + \{B\}^T [R_2][\omega_2][l_2][0 \mid 1]^T\end{aligned} \quad (54)$$

$$\begin{aligned}\dot{C}_1 = \{\dot{C}\}^T [R_3][l_3][0 \mid 1]^T + ...\\ + \{C\}^T [R_3][\omega_3][l_3][0 \mid 1]^T\end{aligned} \quad (55)$$

$$\{\dot{B}\}^T = \{\dot{C}\}^T = \{\dot{r}_{A_2}\}^T - \{\dot{r}_{A_3}\}^T \quad (56)$$

$$\{\dot{r}_{A_2}\}^T = \{l_2\}^T [\omega_2]^T [R_2]^T \quad (57)$$

$$\{\dot{r}_{A_3}\}^T = \{l_3\}^T [\omega_3]^T [R_3]^T \quad (58)$$

4. ESTABLISHMENT OF THE EQUATIONS OF MOTION FOR THE MECHANICAL SYSTEM IN PRESENCE OF CONSTRAINTS

In the presence of constraints the differential equations which describe the motion of the mechanical system may be written as followings:

$$[J]\{\dot{\omega}\} = \{M_m\} + \{M_r\} - [K]\{\varphi\} + [L_\lambda]^T \lambda \quad (59)$$

In the relation (59) the terms involved have the followings expressions:

$$\{\dot{\omega}\} = [\dot{\omega}_1 \mid \dot{\omega}_2 \mid \dot{\omega}_3 \mid \dot{\omega}_4]^T \quad (60)$$

$$\{\varphi\} = [\varphi_1 \mid \varphi_2 \mid \varphi_3 \mid \varphi_4]^T \quad (61)$$

$$[J] = \begin{bmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & J_3 & 0 \\ 0 & 0 & 0 & J_4 \end{bmatrix} \quad (62)$$

$$\{M_m\} = [M_m \mid 0 \mid 0 \mid 0]^T \quad (63)$$

$$\{M_r\} = [0 \mid 0 \mid 0 \mid -M_r]^T \quad (64)$$

$$\{M_r\} = -[C]\{\omega\} \quad (65)$$

$$[L_\lambda] = [0 \mid B_1 \mid -C_1 \mid 0] \quad (66)$$

$$[C] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4 \end{bmatrix} \quad (67)$$

$$[K] = \begin{bmatrix} [K_{12}] & [0] \\ [0] & [K_{34}] \end{bmatrix} \quad (68)$$

$$[K_{12}] = \begin{bmatrix} -k_{12} & k_{12} \\ k_{12} & -k_{12} \end{bmatrix} \quad (69)$$

$$[K_{34}] = \begin{bmatrix} -k_{34} & k_{34} \\ k_{34} & -k_{34} \end{bmatrix} \quad (70)$$

$$[0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (71)$$

In relation (59) "λ" represents Lagrange multiplier which is unknown for now.

5. REMOVAL THE CONTACT FORCES AND ESTABLISHING THE SYSTEM OF DIFFERENTIAL EQUATIONS WHICH DESCRIBES THE MOTION OF MECHANICAL SYSTEM

By multiplying to the left of the matrix relation (59) with the matrix $[L_\tau]^T$ we will obtain the following system of differential equations:

$$[\tilde{J}]\{\dot{\omega}\} = \{\tilde{M}_m\} + \{\tilde{M}_r\} - [\tilde{K}]\{\varphi\} \quad (72)$$

In the relation (72) the terms involved have the followings expressions:

$$[\tilde{J}] = \begin{bmatrix} [L_\tau]^T [J] \\ [0 \mid B_1 \mid -C_1 \mid 0] \end{bmatrix} \quad (73)$$

$$\{\tilde{M}_m\} = \left[\{M_m\}^T [L_\tau] \mid 0\right]^T \quad (74)$$

$$\{\tilde{M}_r\} = -[\tilde{C}]\{\omega\} \tag{75}$$

$$[\tilde{C}] = \left[\begin{array}{c} [L_\tau]^T [C] \\ \hline [0 \mid -B_1 \mid C_1 \mid 0] \end{array}\right] \tag{76}$$

$$[\tilde{K}] = \left[\begin{array}{c} [L_\tau]^T [K] \\ \hline [0 \mid 0 \mid 0 \mid 0] \end{array}\right] \tag{77}$$

$$[L_\tau]^T = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \hline 0 & C_1 & B_1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{78}$$

In the literature the matrix $[L_\tau]$ is called orthogonal complement. The system of differential equations (72) will be solved by using numerical integration methods and the results will be obtained.

6. SOLVING THE SYSTEM OF DIFFERENTIAL EQUATIONS AND GETTING RESULTS

The system of differential equations (72) will be solved for two particular cases.
Each particular case corresponds to a set of values for the mechanical characteristics.
In the first case, the input data are the followings:

$$M_m = 100 - 20 \cdot \omega_1 - \omega_1^2 \ [Nm] \tag{79}$$

$$M_r = 20 - 0.05 \cdot \omega_1 \ [Nm] \tag{80}$$

Mechanical end geometrical characteristics of the system are the followings:

$$l = 0.5\,[m] \tag{81}$$

$$l_2 = 0.05\,[m] \tag{82}$$

$$l_3 = 0.5\,[m] \tag{83}$$

$$J_1 = 0.5\ [kg \cdot m^2] \tag{84}$$

$$J_2 = 0.2\ [kg \cdot m^2] \tag{85}$$

$$J_3 = 0.3\ [kg \cdot m^2] \tag{86}$$

$$J_4 = 0.7\ [kg \cdot m^2] \tag{87}$$

Elastic characteristics for those two linear springs that compound the system are:

$$k_{12} = 100\,[N \cdot m] \tag{88}$$

$$k_{34} = 100\,[N \cdot m] \tag{89}$$

In the second case, the input data are almost the same as in the first case .The only difference between the two sets of input data is related to the values of the two elastic constants of the two springs:

$$k_{12} = 100\,[N \cdot m] \tag{90}$$

$$k_{34} = 40\,[N \cdot m] \tag{91}$$

In the first case will get the results shown in figures 2 and 3.(fig.2, fig.3). In the second case the results presented in the figures 4 and 5 will be get.(fig.4 and fig.5).

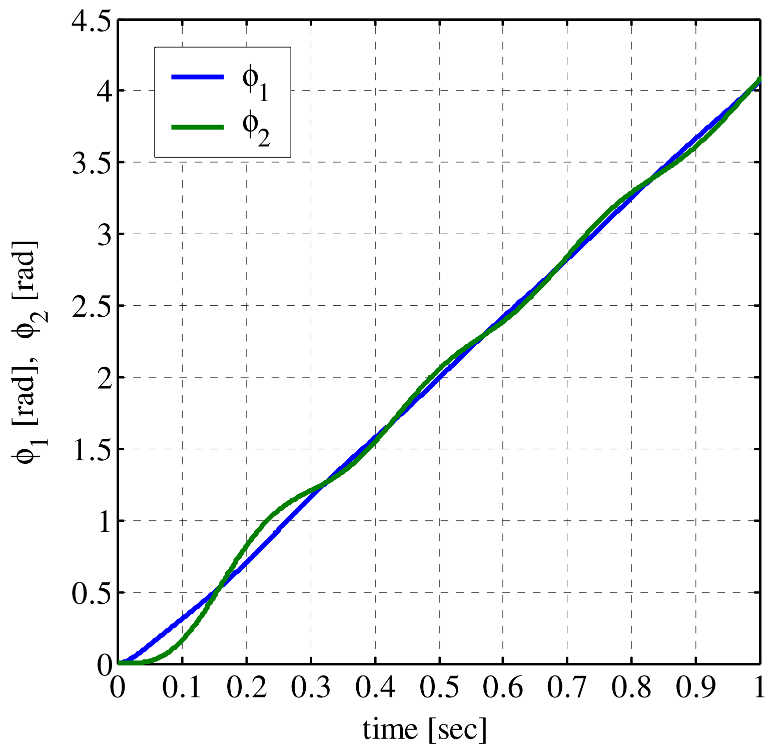

Fig.2 Variation of angular measurements φ_1 and φ_2 as function of time

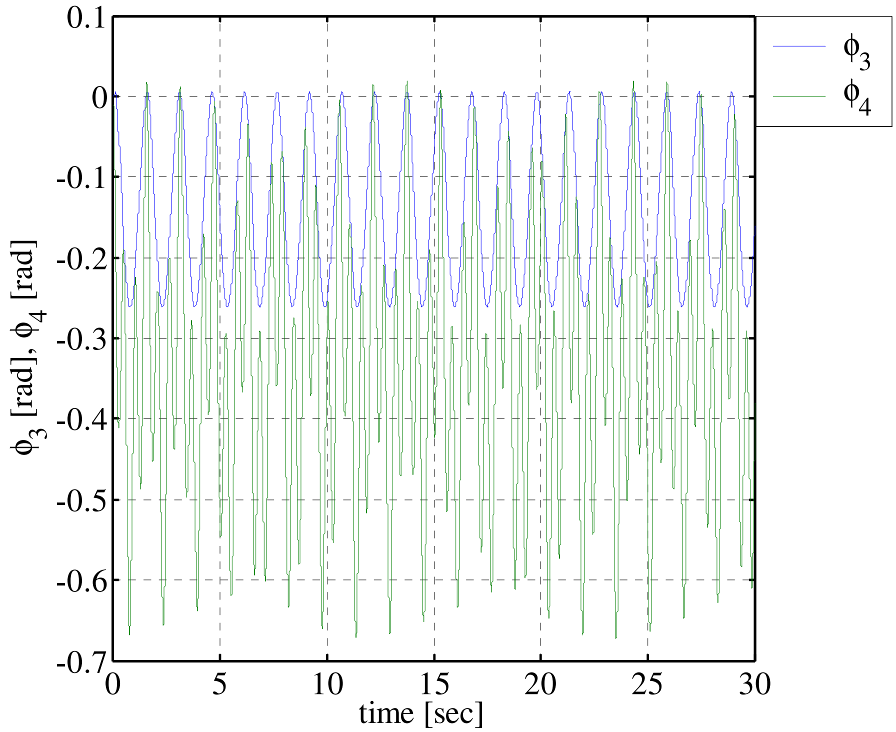

Figure 3 Variation of angular measurements φ_3 and φ_4 as function of time

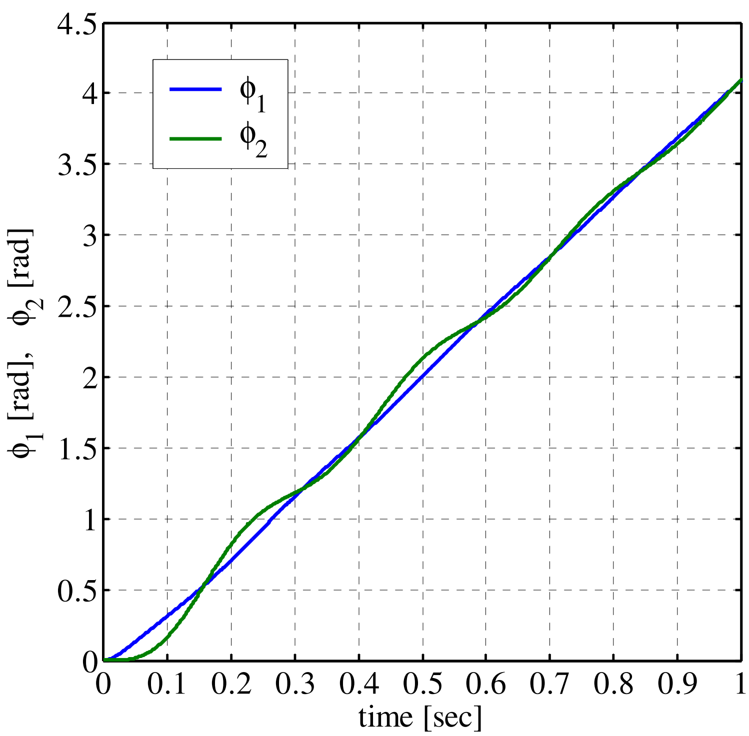

Figure 4 Variation of angular measurements φ_1 and φ_2 as function of time

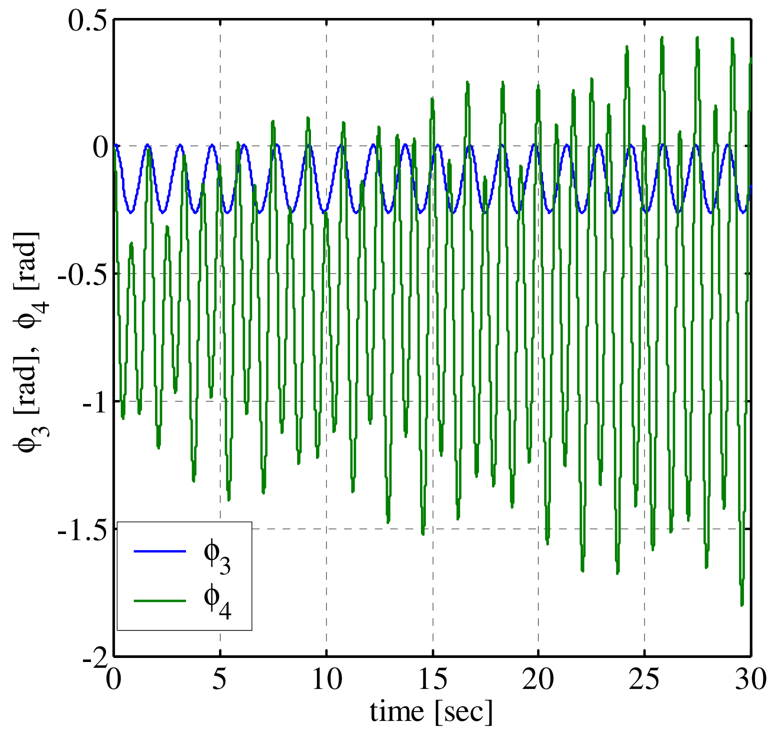

Figure 5 Variation of angular measurements φ_3 and φ_4 as function of time

7. CONCLUSIONS

In the paper was presented a numerical method used to analyze the dynamics of a cyclic mechanism consisting of four solid rigid bodies and linear elastic couplings. This may be considered a mechanical system with constraints. In the case of mechanical systems with constraints related forces that occur in all cases are unknown and consequently must be eliminated. The mathematical model presented in this paper aims to determine the motion of the system for two particular cases. Each particular case corresponds to a set of values of the elastic constants of the two elastic couplings. Mechanical system considered in the paper is only an example. Dynamic study method presented in the paper may used for dynamic survey of any other mechanical system.

8. REFERENCES

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