A NUMERICAL METHOD USED TO ANALYZE THE DYNAMICS OF A MECHANICAL SYSTEM WITH TWO DEGREES OF FREEDOM

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Abstract. *The paper presents a numerical method used to analyze the dynamics of a mechanical system that presents two inner links namely: one elastic link by a non- linear spring and one link by made with the help of a gearing. The whole system has two degrees of freedom.*

Keywords: numerical method, mechanical system, inner link, gearing

1. INTRODUCTION

We will consider the mechanical system which is shown in the figure below (fig.1). The mechanical system consists of an electric engine, a gear reducer and a working machine. The electric engine develops the motor torque M_m. The third element of the system is acted upon by the resistant torque M_r.

2. DIFFERENTIAL EQUATIONS ESTABLISHMENT FOR EACH ELEMENT OF THE SYSTEM WHICH IS CONSIDERED TO BE FREE

The differential equation that describes the movement of the first element of the system may be written as followings:

$$J_1 \cdot \dot{\omega}_1 = M_m - K_{12} \cdot \Delta\varphi_{12} \qquad (1)$$

where:

$$\Delta\varphi_{12} = \varphi_2 - \varphi_1 \qquad (2)$$

$$K_{12} = K_1 + k_1 \cdot (\varphi_2 - \varphi_1)^2 \qquad (3)$$

In relation (3) "k_{12}" represents the linear elastic characteristic of the spring.

The differential equation that describes the movement of the second element of the system may be written as followings:

$$J_2 \cdot \dot{\omega}_2 = K_{12} \cdot \Delta\varphi_{12} \qquad (4)$$

The differential equation that describes the movement of the third element of the system may be written as followings:

$$J_3 \cdot \dot{\omega}_3 = -c_3 \cdot \omega_3 = M_r \qquad (5)$$

In relation (5) "c_3" represents viscous damping coefficient.

In relations (1), (4) and (5), "J_1", "J_2" and "J_3" represents the mechanical moments of inertia in relation to the axes of rotation of each solid rigid which made up the syste

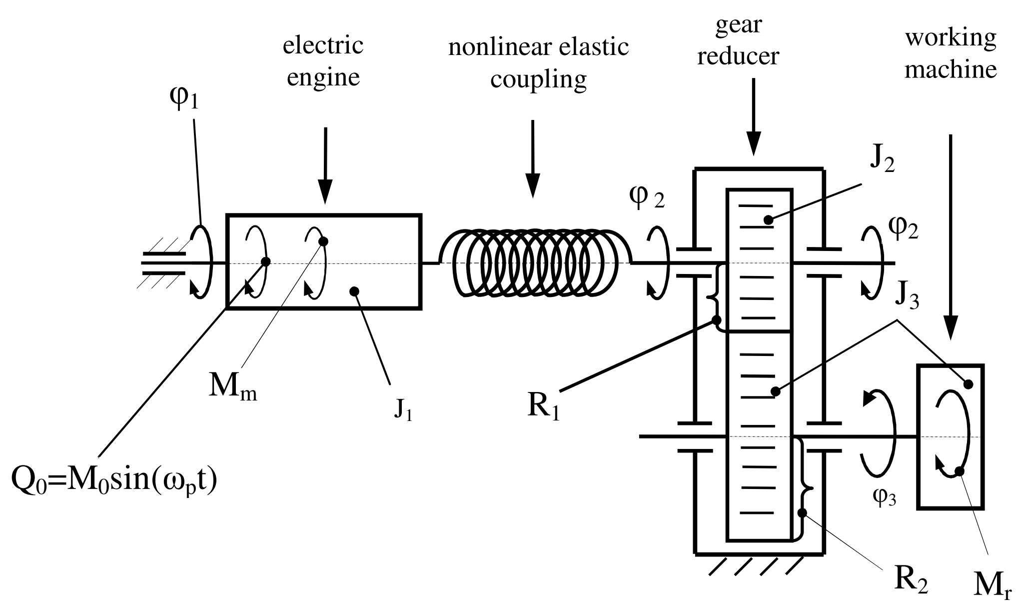

Figure 1. Mechanical system consisting of an electric engine, gear reducer and a working machine

3. ESTABLISHMENT OF THE RELATIONSHIPS BETWEEN KINEMATICAL PARAMETERS OF THE SOLID RIGIDS THAT COMPOUND THE SYSTEM

The relationship between kinematical parameters of the rigid solids “2” and “3” may be written as followings:

$$R_1 \cdot \varphi_2 + R_2 \cdot \varphi_3 = 0 \tag{6}$$

Under differential form the above relationship may be written as follows:

$$R_1 \cdot \dot{\varphi}_2 + R_2 \cdot \dot{\varphi}_3 = 0 \tag{7}$$

or:

$$R_1 \cdot \omega_2 + R_2 \cdot \omega_3 = 0 \tag{8}$$

where:

$$\dot{\varphi}_1 = \omega_1 \tag{9}$$

$$\dot{\varphi}_2 = \omega_2 \tag{10}$$

4. ESTABLISHMENT OF THE DIFFERENTIAL EQUATIONS SYSTEM THAT DESCRIBES THE MOTION OF THE SYSTEM IN THE PRESENCE OF CONSTRAINTS

If we calculate the derivative of the relation (8) with respect to time we will obtain:

$$R_1 \cdot \dot{\omega}_2 + R_2 \cdot \dot{\omega}_3 = 0 \tag{11}$$

The differential equations system that describes the motion of the system may be written as followings:

$$[J]\{\dot{\omega}\} = \{Q\} + [C]\{\omega\} + [K]\{\varphi\} + [L_\lambda]^T \lambda \tag{12}$$

The terms involved in the relationship (12) have the followings expressions:

$$[J] = \left[\begin{array}{c|c|c} J_1 & 0 & 0 \\ \hline 0 & J_2 & 0 \\ \hline 0 & 0 & J_3 \end{array}\right] \tag{13}$$

$$\{\dot{\omega}\} = [\dot{\omega}_1 \mid \dot{\omega}_2 \mid \dot{\omega}_3]^T \tag{14}$$

$$\{\omega\} = [\omega_1 \mid \omega_2 \mid \omega_3]^T \tag{15}$$

$$\{Q\} = \{Q_0\} + \{Q_m\} \tag{16}$$

$$\{Q_0\} = \left[M_0 \sin(\omega_p t) \mid 0 \mid 0\right]^T \tag{17}$$

$$\{Q_m\} = [M_m \mid 0 \mid 0]^T \tag{18}$$

$$\{\varphi\} = [\varphi_1 \mid \varphi_2 \mid \varphi_3]^T \tag{19}$$

$$Mm = A - B\omega_1 - C\omega_1^2 \tag{20}$$

$$[K] = \left[\begin{array}{c|c|c} -K_{12} & K_{12} & 0 \\ \hline K_{12} & -K_{12} & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \tag{21}$$

$$[C] = -\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & c \end{array}\right] \tag{22}$$

$$[L_\lambda] = [0 \mid R_1 \mid R_2] \tag{23}$$

Taking into account the relationships of connection the system of differential equations may be written as:

$$[J_1]\{\dot{\omega}\} = \{Q_1\} + [C_1]\{\omega\} + [K_1]\{\varphi\} \tag{24}$$

The terms involve in the relationship (23) have the followings expressions:

$$[J_1] = \left[\begin{array}{c} [L_\tau]^T [J] \\ \hline [0 \mid J_2 R_1 \mid J_2 R_2] \end{array}\right] \tag{25}$$

$$\{Q\}_1 = \{Q_0^1\} + \{Q_m^1\} \tag{26}$$

$$\{Q_0^1\} = [L_\tau]^T \{Q_0\} \tag{27}$$

$$\{Q_m^1\} = [L_\tau]^T \{Q_m\} \tag{28}$$

$$[C_1] = \left[\begin{array}{c} [L_\tau]^T [C] \\ \hline [0 \mid 0 \mid 0] \end{array}\right] \tag{29}$$

$$[K_1] = \left[\begin{array}{c} [L_\tau]^T [K] \\ \hline [0 \mid 0 \mid 0] \end{array}\right] \tag{30}$$

$$[L_\tau]^T = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & -(R_1/R_2) \end{array}\right] \tag{31}$$

The second-order differential equations system (24) can be transformed into one of the first order by entering the following notation:

$$\{\dot{\varphi}\} = \{\omega\} \tag{32}$$

where:

$$\{\dot{\varphi}\} = [\dot{\varphi}_1 \mid \dot{\varphi}_2 \mid \dot{\varphi}_3]^T \tag{33}$$

Relationships (24) and (32) can be written together as followings:

$$[J_2]\{\dot{\alpha}\} = \{Q_2\} + [C_2]\{\alpha\} + [K_2]\{\alpha\} \tag{34}$$

where:

$$[J_2] = \left[\begin{array}{c|c} [J_1] & [0] \\ \hline [0] & [I_3] \end{array}\right] \tag{35}$$

$$[I_3] = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \quad (36)$$

$$[0] = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \quad (37)$$

$$[C_2] = \left[\begin{array}{c|c} [C_1] & [0] \\ \hline [0] & [0] \end{array}\right] \quad (38)$$

$$[K_2] = \left[\begin{array}{c|c} [0] & [K_1] \\ \hline [I_3] & [0] \end{array}\right] \quad (39)$$

$$\{Q_2\} = \left[\{Q_1\}^T \mid \{0\}^T\right]^T \quad (40)$$

$$\{\dot{\alpha}\} = \left[\{\dot{\omega}\}^T \mid \{\dot{\varphi}\}^T\right]^T \quad (41)$$

$$\{\alpha\} = \left[\{\omega\}^T \mid \{\varphi\}^T\right]^T \quad (42)$$

The generalized system of differential equations may be written as followings:

$$[J_3]\{\dot{\beta}\} = \{Q_3\} + [C_3]\{\beta\} + [K_3]\{\beta\} \quad (43)$$

where:

$$[J_3] = \left[\begin{array}{c|c|c} [J_1] & [0] & [0] \\ \hline [0] & [I_3] & [0] \\ \hline [0] & [0] & [I_3] \end{array}\right] \quad (44)$$

$$\{\dot{\beta}\} = \left[\{\dot{\alpha}\}^T \mid \{\dot{F}\}^T\right]^T \quad (45)$$

$$\{\beta\} = \left[\{\alpha\}^T \mid \{F\}^T\right]^T \quad (46)$$

$$\{\dot{F}\} = [\dot{F}_1 \mid \dot{F}_2 \mid 0]^T \quad (47)$$

$$\{F\} = [F_1 \mid F_2 \mid 0]^T \quad (48)$$

$$F_1 = -K_{12} \cdot \Delta\varphi_{12} \quad (49)$$

$$F_2 = K_{12} \cdot \Delta\varphi_{12} \quad (50)$$

$$F_3 = 0 \quad (51)$$

$$\dot{F}_1 = dF_1/dt = -\overset{\bullet}{(K_{12}\Delta\varphi_{12})} \quad (52)$$

$$\dot{F}_1 = -\left[K_1 + 3k_1(\Delta\varphi_{12})^2\right] \cdot \omega_{21} \quad (53)$$

$$\dot{F}_2 = dF_2/dt = \overset{\bullet}{(K_{12}\Delta\varphi_{12})} \quad (54)$$

$$\dot{F}_2 = \left[K_1 + 3k_1(\Delta\varphi_{12})^2\right] \cdot \omega_{21} \quad (55)$$

$$\dot{F}_3 = dF_3/dt = 0 \quad (56)$$

$$\omega_{21} = \omega_2 - \omega_1 \quad (57)$$

$$\{Q_3\} = \left[\{Q_2\}^T \mid \{0\}^T\right]^T \quad (58)$$

$$\{0\} = [0 \mid 0 \mid 0]^T \quad (59)$$

$$[C_3] = \left[\begin{array}{c} [C_2] \\ \hline [[0] \mid [0]] \end{array}\right] \quad (60)$$

$$[K_3] = \left[\begin{array}{c} [K_2] \\ \hline [[K_4] \mid [0]] \end{array}\right] \quad (61)$$

$$[K_4] = d([K_1])/d(\Delta\varphi_{12}) \quad (62)$$

$$[K_4] = \left[\begin{array}{c|c|c} K_4^{11} & K_4^{12} & K_4^{13} \\ \hline K_4^{21} & K_4^{22} & K_4^{23} \\ \hline K_4^{31} & K_4^{32} & K_4^{33} \end{array}\right] \quad (63)$$

$$K_4^{11} = -\left[K_1 + 3k_1(\Delta\varphi_{12})^2\right] \quad (64)$$

$$K_4^{12} = \left[K_1 + 3k_1(\Delta\varphi_{12})^2\right] \quad (65)$$

$$K_4^{13} = 0 \quad (66)$$

$$K_4^{21} = \left[K_1 + 3k_1(\Delta\varphi_{12})^2\right] \quad (67)$$

$$K_4^{22} = -\left[K_1 + 3k_1(\Delta\varphi_{12})^2\right] \quad (68)$$

$$K_4^{23} = 0 \quad (69)$$

$$K_4^{31} = 0 \quad (70)$$

$$K_4^{32} = 0 \quad (71)$$

$$K_4^{33} = 0 \quad (72)$$

Matrix $[L_\tau]$ which is expressed by relationship (31) is known in the literature as natural orthogonal complement.

5. METHOD OF DETERMINATION OF THE NATURAL ORTHOGONAL COMPLEMENT

In this paragraph a method of determination of the natural orthogonal complement will be presented.
First of all we shall write relation (8) under matrix form as followings:

$$[L_\lambda]\{\omega\} = 0 \quad (73)$$

The terms involved in relationship (73) have the followings expressions:

$$[L_\lambda] = [0 \mid R_1 \mid R_2] \quad (74)$$

$$\{\omega\} = [\omega_1 \mid \omega_2 \mid \omega_3]^T \quad (75)$$

Relationship (73) may by written under equivalent form as followings:

$$\left[L_\lambda^{ext.}\right]\{\omega\} = [l_\lambda]\{\dot{q}\} \quad (76)$$

The terms involved in relation (76) hav e the following expressions:

$$\{\dot{q}\} = [\omega_1 \mid \omega_2]^T \tag{77}$$

$$\left[L_{\lambda}^{ext.}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & R_1 & R_2 \end{bmatrix} \tag{78}$$

$$[l_{\lambda}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T \tag{79}$$

From relationship (77) one may deduce:

$$\{\omega\} = \left[L_{\lambda}^{ext.}\right]^{-1}[l_{\lambda}]\{\dot{q}\} \tag{80}$$

In relation (80) the following notation may be introduced:

$$[L_{\tau}] = \left[L_{\lambda}^{ext.}\right]^{-1}[l_{\lambda}] \tag{81}$$

Using relationship (81) relationship (80) may be written under the following form:

$$\{\omega\} = [L_{\tau}]\{\dot{q}\} \tag{82}$$

Matrix $[L_{\tau}]$ determined by the relationship (81) is called in literature natural orthogonal complement.

6. DETERMINATION OF OTHERS ORTHOGONAL COMPLEMENTS THAN NATURAL ONES

Dynamic survey of the mechanical system which is shown in fig.1 may be performed using another orthogonal complement than the natural one. For instance the following orthogonal complements may be used:

$$[L_{\tau}]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R_2 & -R_1 \end{bmatrix} \tag{82}$$

or:

$$[L_{\tau}]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -R_2 & +R_1 \end{bmatrix} \tag{83}$$

In both cases we will get the same results.

7. SOLVING THE SYSTEM OF DIFFERENTIAL EQUATIONS AND GETTING RESULTS

The system of differential equations (43) will be solved using numerical integration methods and the results presented in the figures (2…5) will be obtained.

Two cases will be taken into consideration. In the first case the excitatory couple will be neglected. In the second case the excitatory couple will be considered non-zero.

In the first case, the numerical input data are the followings:

$$J_1 = 10\ kg \cdot m^2 \tag{84}$$

$$J_2 = J_3 = 5\ kg \cdot m^2 \tag{85}$$

J_1, J_2 and J_3 represent the mechanical inertia moments of the three rotating rigid solids.

$$K_1 = 100\ N \cdot m \tag{86}$$

$$k_1 = 10\ N \cdot m \tag{87}$$

$$A = 200\ N \cdot m \tag{88}$$

$$B = 10\ N \cdot m \cdot s \tag{89}$$

$$C = 1\ N \cdot m \cdot s^2 \tag{90}$$

$$c = 100\ N \cdot m \cdot s \tag{91}$$

c - represents viscous damping coefficient

$$M_0 = 0\ N \cdot m \tag{92}$$

M_0 represents excitatory torque amplitude.

$$\omega_p = 0\ s^{-1} \tag{93}$$

ω_p represents excitatory torque pulsation

$$t = 10\ sec. \tag{94}$$

t – represents the duration of motion

$$R_1 = 2\ [meters] \tag{95}$$

$$R_2 = 3\ [meters] \tag{96}$$

In the first case, in the absence of harmonic torque fluctuations of we will obtain the results shown in fig.2 and fig.3.

In the second case, the numerical input data are the followings:

$$J_1 = 10\ kg \cdot m^2 \tag{97}$$

$$J_2 = J_3 = 5\ kg \cdot m^2 \tag{98}$$

J_1, J_2 and J_3 represent the mechanical inertia moments of the three rotating rigid solids.

$$K_1 = 100\ N \cdot m \tag{99}$$

$$k_1 = 10\ \mathrm{N \cdot m} \tag{100}$$

$$A = 200\ \mathrm{N \cdot m} \tag{101}$$

$$B = 10\ \mathrm{N \cdot m \cdot s} \tag{102}$$

$$C = 1\ \mathrm{N \cdot m \cdot s^2} \tag{103}$$

$$c = 100\ \mathrm{N \cdot m \cdot s} \tag{104}$$

c represents viscous damping coefficient

$$M_0 = 20\ \mathrm{N \cdot m} \tag{105}$$

M_0 represents excitatory torque amplitude.

$$\omega_p = 5\ \mathrm{sec}^{-1} \tag{106}$$

ω_p represents excitatory torque pulsation

$$t = 10\ \mathrm{sec}. \tag{107}$$

t – represents duration of motion

$$R_1 = 2\ [\mathrm{meters}] \tag{108}$$

$$R_2 = 3\ [\mathrm{meters}] \tag{109}$$

In the second case, in the presence of harmonic torque fluctuations we will obtain the results shown in fig.4 and fig.5.

Figures in the phase plane

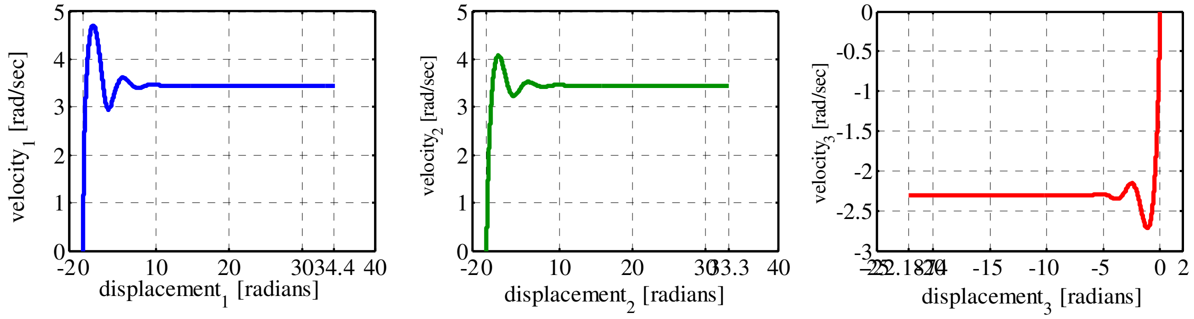

Fig.2 Figures in the phase plane

Variation of elastic moments with respect to time

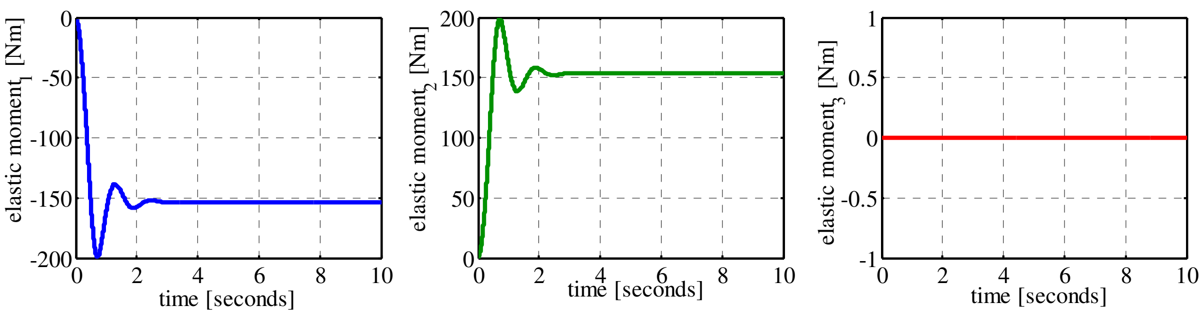

Fig.3 Variation of elastic moments with respect to time

Figures in the phase plane

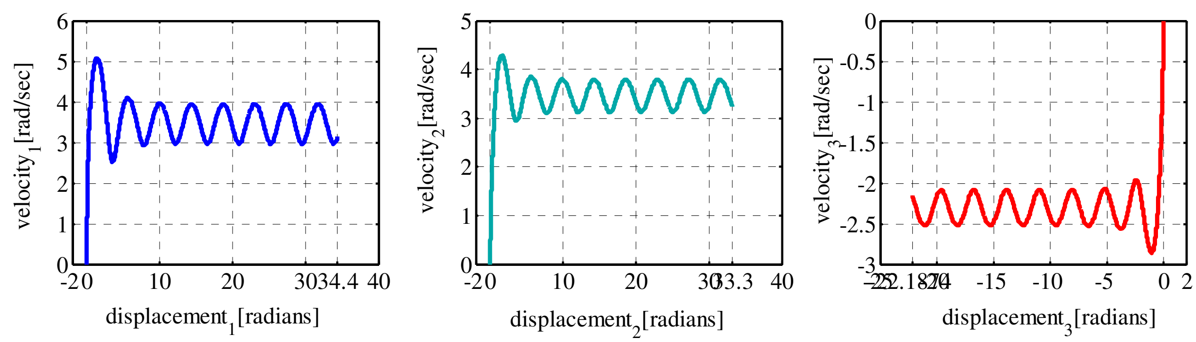

Fig.4 Figures in the phase plane

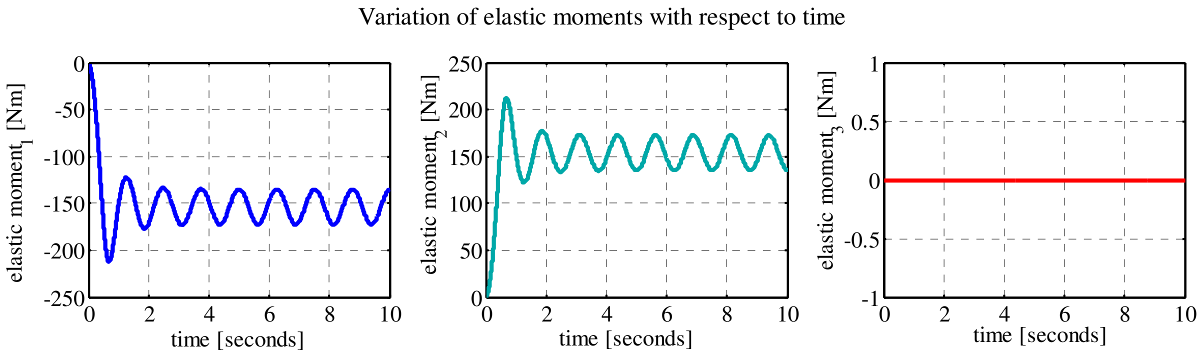

Fig.5 Variation of elastic moments with respect to time

In the absence of harmonic torque fluctuations it may be observed a stabilization of a uniform motion. In the presence of harmonic torque fluctuations it may be observed an oscillatory motion which accompanies the uniform motion. Angular speed oscillations take place around the value of constant angular speed.

8. CONCLUSIONS

In the paper was presented a numerical method used to analyze the dynamics of a mechanical system consisting of an electric engine, a nonlinear elastic coupling, a gear reducer and a working machine.
In the case of mechanical systems with constraints related forces that occur in all cases are unknown and consequently must be eliminated.
The mathematical model presented in this paper aims to determine both the movement and variation in relation to time of the moments produced by those two nonlinear elastic couplings.
The mechanical system considered in the paper is only an example. Dynamic study method presented in the paper may used for dynamic survey of any other mechanical system.

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