SECTIONAL FORCES DIAGRAMS IN POLAR COORDINATES FOR CIRCULAR CANTILEVERS USING MATHCAD (I)

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Abstract: *The sectional forces diagrams in polar coordinates for circular beams can be plotted using the step-function available in MATHCAD (2011). The suggested method has the advantage of allowing a fast identification of the critical sections subjected to bending and the position of concentrated loads. The step-function Φ allows an uniform and consistent expression and representation of the functions of the sectional forces in polar coordinates. This present paper deals with the method of determination of the analytical functions. Two particular examples for the determination of sectional forces diagrams for circular cantilevers under radial loading will also be shown.*

Keywords: *diagrams, polar coordinates, Mathcad, circular cantilever*

1. PROBLEM DEFINITION

The beam *AB* (Fig. 1) is defined by a circular geometric axis is a circular cantilever with its free end in section A and its fixed end in section B. The variable central angle is denoted by α. The cantilever is loaded with an uniformly distributed radial load q on the length *AE* (having the central angle β with respect to point A), a concentrated force P and Q in section *D* (with the variable central angle φ) and the moment M_0 in section *G* (with the variable central angle ψ).

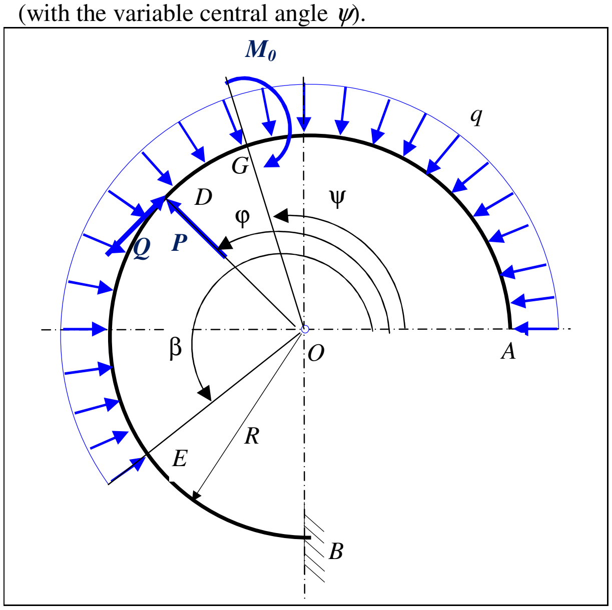

Fig.1: General layout of the circular cantilever

Requested tasks:

1. Find the general analytical expressions of the axial force $N(\theta)$, shear force $T(\theta)$ and bending moment $Mi(\theta)$ as a function of the uniformly distributed radial load q.
2. Find the differential relations of the axial force, shear force and bending moment depending on the uniformly distributed radial load q.
3. Find the force-couple system in section E corresponding to the exterior forces and the general expressions of the reaction forces in section B.
4. Plot the diagrams of the axial force $N(\theta)$, shear force $T(\theta)$ and bending moment $Mi(\theta)$ using the step-function in MATHCAD.

1.1. General analytical expressions of the axial force, shear force and bending moment

In order to determine the general expressions of the sectional forces corresponding to the uniformly distributed radial load q, a beam element will be considered having the length ds , located at an angular distance ε from the free end of the circular cantilever (Fig.2). The corresponding elementary force will be:

$$dF = q \cdot ds = q \cdot R \cdot d\varepsilon$$

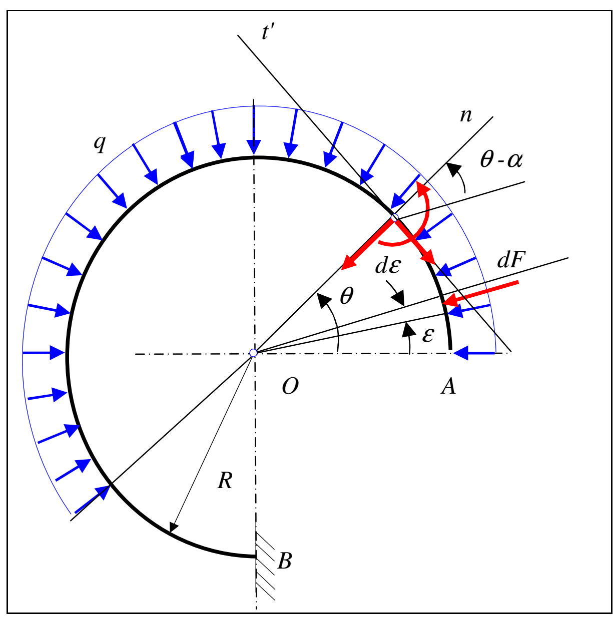

Fig. 2: Determination of the sectional forces by the integration of the elementary force *dF*

The analytical expressions of the axial and shear sectional forces, $N(\theta)$ and $T(\theta)$, can be obtained using the uniformly distributed load on the sector *AE*, by integrating the projection of the elementary force dF on the normal and tangential directions, *On* and *tt'* respectively (Fig. 2).
If we consider the same sign convention as in the case of straight beams (Fig.3), the following expressions will be obtained [3- Marin C, 2012]:

$$\begin{cases} N(\theta)=-\int_0^{\theta}(qR\cdot d\varepsilon)\cdot \sin(\theta-\varepsilon)=-qR(1-\cos\theta); \\ T(\theta)=\int_0^{\theta}(qR\cdot d\varepsilon)\cdot \cos(\theta-\varepsilon)=qR\sin\theta; \end{cases} \quad (1)$$

The bending moment $M_i(\theta)$ for an uniformly distributed load q will be obtained for the sector *AE* by calculating the moment of the elementary force dF with respect with the current section and integrating along the arc θ [3- Marin C, 2012]:

$$M(\theta)=-\int_0^{\theta}R\sin(\theta-\varepsilon)\cdot(qR\cdot d\varepsilon)=-qR^2(1-\cos\theta). \quad (2)$$

1.2. Differential relations between the sectional forces and the uniformly distributed radial load *q*
Between the sectional forces $N(\theta)$, $T(\theta)$ or $M_i(\theta)$ and the exterior load q certain differential relations can be defined. The analytical expressions of the forces can be verified using these relations [1- Marin C, 2006].

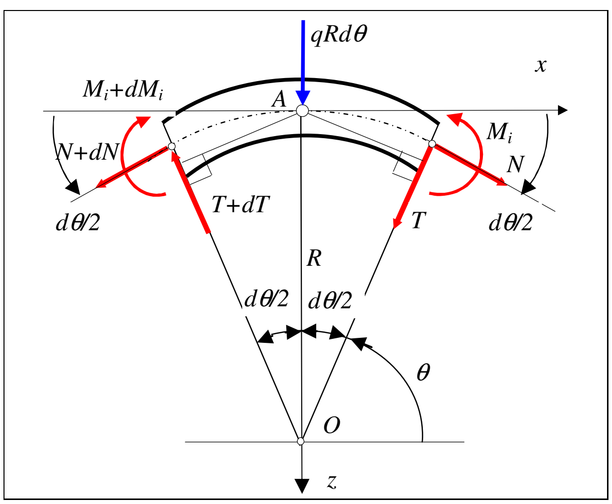

Fig. 3: Beam element for the determination of the differential relations between sectional forces and external loads.

A beam element will be considered (Fig. 3), having the length $ds=R\cdot d\theta$, corresponding to the central angle θ. The element is subjected to the axial forces N and $N+dN$, the shear forces T and $T+dT$ and the bending moments M_i, M_i+dM_i.
The equations of equilibrium between the exterior loads and the sectional forces acting on the ends of the beam element are:

$$\begin{aligned} &\sum F_x=0:\ (-N-dN+N)\cdot\cos\frac{d\theta}{2}-(T+T+dT)\cdot\sin\frac{d\theta}{2}=0 \\ &\sum F_z=0\ :(N+N+dN)\cdot\sin\frac{d\theta}{2}+(-T-dT+T)\cdot\cos\frac{d\theta}{2}+qR\cdot d\theta=0 \\ &\sum M_{Ay}=0:-(T+T+dT)\cdot R\cdot\sin\frac{d\theta}{2}+(N+dN-N)\cdot R\cdot\left(1-\cos\frac{d\theta}{2}\right)-dM_i=0 \end{aligned} \quad (3)$$

The following assumption can be made for very small angles $d\theta$:

$$\sin\frac{d\theta}{2}\cong\frac{d\theta}{2};\quad \cos\frac{d\theta}{2}\cong 1;\quad \left(\frac{d\theta}{2}\right)^2\cong 0 \quad (4)$$

Therefore, the equations system (3) becomes:

$$\begin{cases} -dN-T\cdot d\theta=0 \\ N\cdot d\theta-dT+q\cdot R\cdot d\theta=0 \\ dM_i+T\cdot R\cdot d\theta=0 \end{cases} \quad (5)$$

The differential relations (5) between sectional forces and external loads can be also expressed as [4- Marin C, 2009]:

$$\begin{cases} \dfrac{dN}{d\theta}=-T\ ; \\ \dfrac{dT}{d\theta}=N+qR; \\ \dfrac{dM_i}{d\theta}=-T\cdot R \end{cases} \quad (6)$$

The analytical expressions of forces (1) and bending moment (2) can be verified [4- Marin C, 2009] using the diferential equations (6):

$$\begin{cases} N(\theta)=-qR\cdot(1-\cos\theta) \\ T(\theta)=qR\cdot\sin\theta \\ M_i(\theta)=-qR^2(1-\cos\theta) \end{cases} \Rightarrow \begin{cases} \dfrac{dN}{d\theta}=-qR\cdot\sin\theta; \\ \dfrac{dT}{d\theta}=qR\cdot\cos\theta \\ \dfrac{dM_i}{d\theta}=-qR^2\cdot\sin\theta \end{cases} \quad (7)$$

1.3. The expressions of the equivalent force-couple system in section E and the reaction forces
The equivalent force-couple system in section E (Fig. 4) corresponding to the uniformly distributed radial load q can be determined using the expressions of the sectional forces (1) and (2), for the particular value of the angle: $\theta=\beta$ [1- Marin C, 2006]:

$$\tau_E:\ \begin{cases} N_\beta=-qR\cdot(1-\cos\beta) \\ T_\beta=qR\cdot\sin\beta \\ M_{i\beta}=-qR^2(1-\cos\beta) \end{cases} \quad (8)$$

The reaction forces in the fixed support B (H_B, V_B, M_B) can be determined using the equivalent force-couple system (1), the force P and the bending moment M_0 [2- Marin C, 2007]:

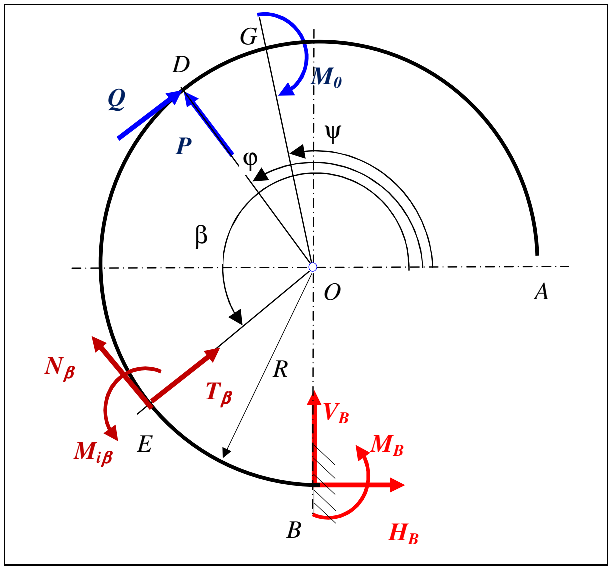

Fig. 4: Determination of the equivalent force-couple system and the reaction forces

$$\begin{cases} H_B = -P\cdot\cos\varphi - Q\sin\varphi - N_\beta\sin\beta + T_\beta\cos\beta \\ \Rightarrow H_B = -P\cdot\cos\varphi + Q\sin\varphi + qR\cdot\sin\beta \\ V_B = -P\cdot\sin\varphi + Q\cos\varphi + N_\beta\cos\beta + T_\beta\sin\beta \\ \Rightarrow V_B = -P\cdot\sin\varphi - Q\cos\varphi + qR(1-\cos\beta) \\ M_B = M_0 + PR\cdot\cos\varphi + QR(1+\sin\varphi) - M_\beta + \\ \quad + N_\beta R\cdot(1+\sin\beta) - T_\beta R\cdot\cos\beta \\ \quad \Rightarrow M_B = M_0 + PR\cdot\cos\varphi + QR(1+\sin\varphi) - q\cdot R^2\cdot\sin\beta \end{cases} \tag{9}$$

1.4. MATHCAD functions for plotting the forces diagrams in polar coordinates

The forces diagrams in polar coordinates can be plotted using the step-function Φ, available in Mathcad [5-Mathcad, 2011]. Their geometrical axis will be a circle of radius $R' = 10\,R$.

The same sign convention as in the case of straight beams will be adopted: positive N and T are represented towards the exterior of the geometry axis and positive M_i towards the interior.

The diagrams in the sections corresponding to the concentrated loads are characterized by jumps and therefore the limit values of the sectional forces will be determined at the left and right of theses sections.

The analytical expressions of the sectional forces can be expressed in polar coordinates using the step-function Φ available in Mathcad [3- Marin C, 2012]:

$$\begin{cases} N(\theta) = 10\cdot R + \left(-qR\cdot(1-\cos\theta)\right)\left(\Phi(\theta)-\Phi(\theta-\beta)\right) + \\ \quad + \left(N_\beta\cos(\theta-\beta) - T_\beta\sin(\theta-\beta)\right)\left(\Phi(\theta-\beta)-\Phi(\theta-\alpha)\right) + \\ \quad + P\cdot\sin(\theta-\varphi)\cdot\left(\Phi(\theta-\varphi)-\Phi(\theta-\alpha)\right) - \\ \quad - Q\cdot\cos(\theta-\varphi)\cdot\left(\Phi(\theta-\varphi)-\Phi(\theta-\alpha)\right) \\ \\ T(\theta) = 10\cdot R + \left(qR\cdot\sin\theta\right)\left(\Phi(\theta)-\Phi(\theta-\beta)\right) + \\ \quad + \left(N_\beta\sin(\theta-\beta) + T_\beta\cos(\theta-\beta)\right)\left(\Phi(\theta-\beta)-\Phi(\theta-\alpha)\right) - \\ \quad - P\cdot\cos(\theta-\varphi)\cdot\left(\Phi(\theta-\varphi)-\Phi(\theta-\alpha)\right) - \\ \quad - Q\cdot\sin(\theta-\varphi)\cdot\left(\Phi(\theta-\varphi)-\Phi(\theta-\alpha)\right) \\ \\ M(\theta) = 10\cdot PR - qR^2\left(1-\cos\theta\right)\left(\Phi(\theta)-\Phi(\theta-\beta)\right) + \\ + \left(M_\beta - N_\beta R(1-\cos(\theta-\beta)) - T_\beta\sin(\theta-\beta)\right)\left(\Phi(\theta-\beta)-\Phi(\theta-\alpha)\right) + \\ \quad + PR\cdot\sin(\theta-\varphi)\cdot\left(\Phi(\theta-\varphi)-\Phi(\theta-\alpha)\right) - \\ \quad + QR\cdot(1-\cos(\theta-\varphi))\cdot\left(\Phi(\theta-\varphi)-\Phi(\theta-\alpha)\right) \\ \quad - M_0\cdot\left(\Phi(\theta-\psi)-\Phi(\theta-\alpha)\right) \end{cases} \tag{10}$$

2. RESULTS

2.1. Particular case A

The reaction forces are determined for the following particular values of the given parameters:
R=1m; P=1 kN; Q=0; M_0*=1kNm; q=1 kN/m;* $\alpha = 3\pi/2$; $\beta = 5\pi/4$; $\varphi = 3\pi/4$; $\psi = 2\pi/3$,
By replacing these values in (9) the reactions forces will be:

$$\begin{cases} H_B = -P\cdot\cos\varphi + qR\cdot\sin\beta = 0 \\ V_B = -P\cdot\sin\varphi + qR(1-\cos\beta) = 1\ kN \\ M_B = M_0 + PR\cdot\cos\varphi - q\cdot R^2\cdot\sin\beta = 1\ \ kNm \end{cases} \tag{11}$$

The forces diagrams (Fig. 5 – 7) were obtained by introducing the sectional forces functions (10) in Mathcad [5 – MATHCAD 2011].

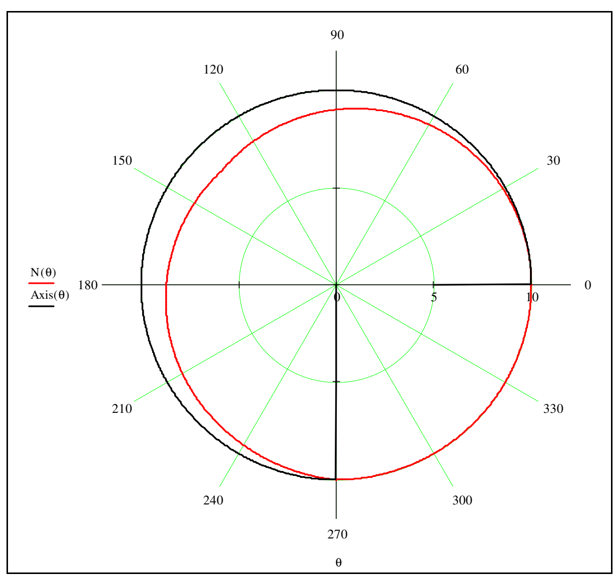

Fig. 5: Axial forces Diagram–case A

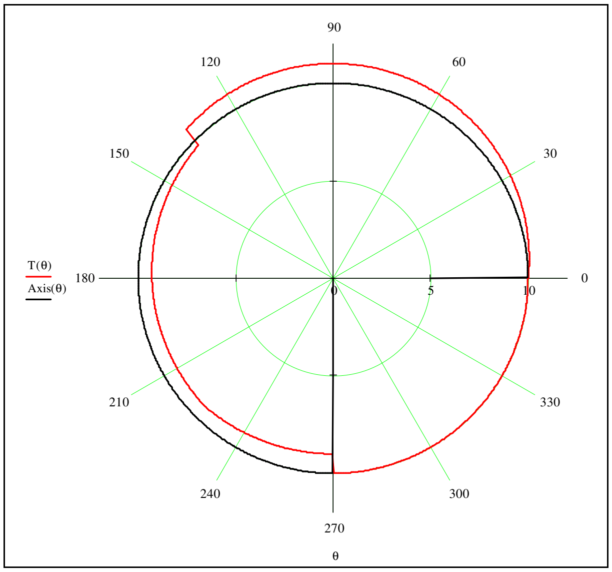

Fig.6: Shear forces diagram–case A

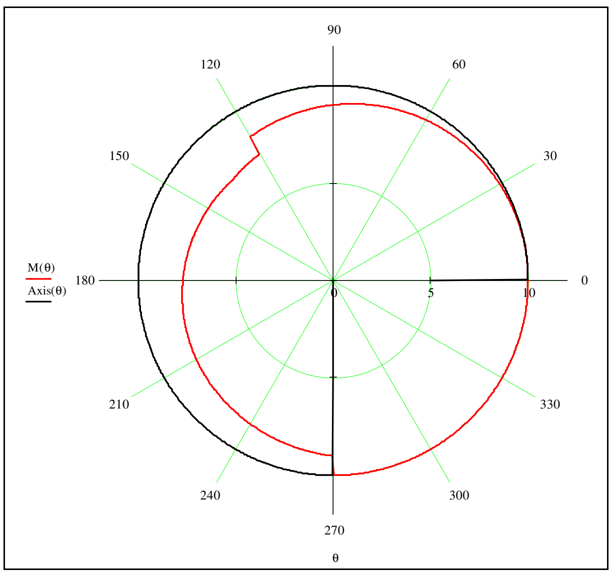

Fig. 7: Bending moments diagram–case A

Conclusion

The particular case A is a general one and allows the identification of jumps in the internal forces diagrams in the sections corresponding to concentrated loads (see Fig. 5 – 7). The diagram values of the reactions in section B correspond with the values determined using equations (9).

2.2. Particular case B

The reaction forces are determined for the following particular values of the given parameters: $R=1m$; $Q=qR$; $M_0=0$; $q=1\ kN/m$; $\alpha=\pi$; $\beta=0$; $\varphi=0$.

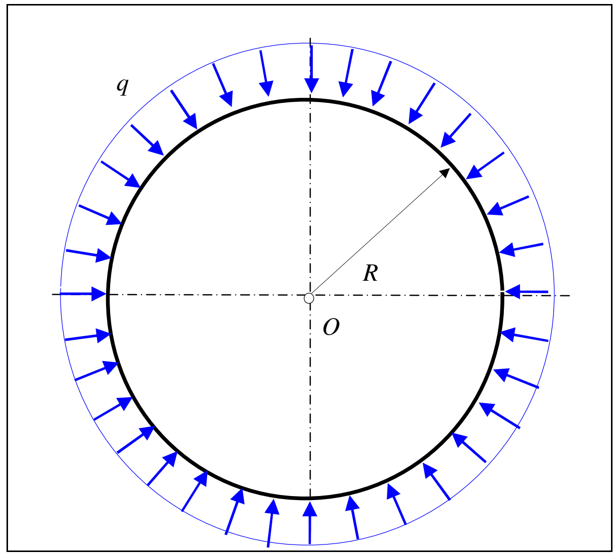

Fig. 8: General layout of the circular cantilever in case B.

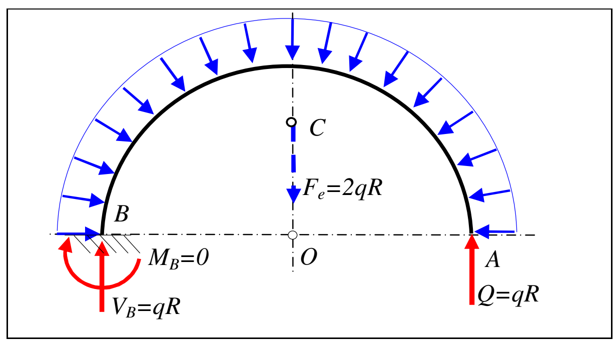

Fig. 9: Particular layout of the semicircular cantilever in case B.

General layout of the circular cantilever - case B (Fig.8) is equivalent with particular layout of the semicircular cantilever (Fig.9)

The reaction forces obtained by replacing the values of the parameters are:

$$\begin{cases} H_B = 0 \\ V_B = qR \\ M_B = 0 \end{cases} \qquad (11)$$

The forces diagrams (Fig. 10 – 12) were obtained by introducing the sectional forces functions (10) in Mathcad [5 – MATHCAD 2011].

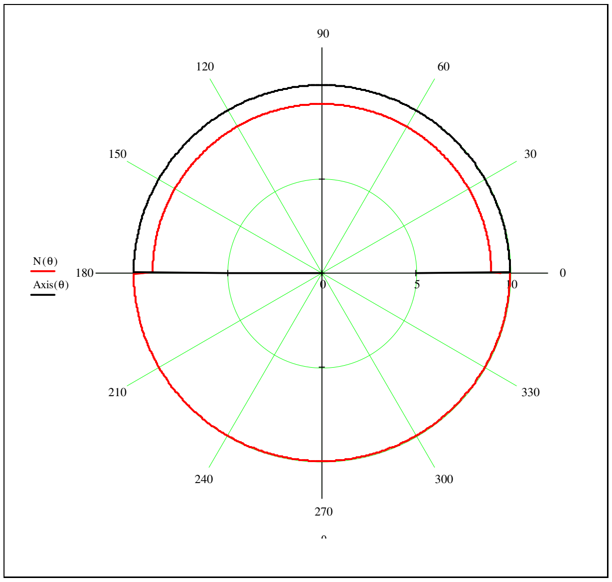

Fig. 10: Axial forces diagram–caseB

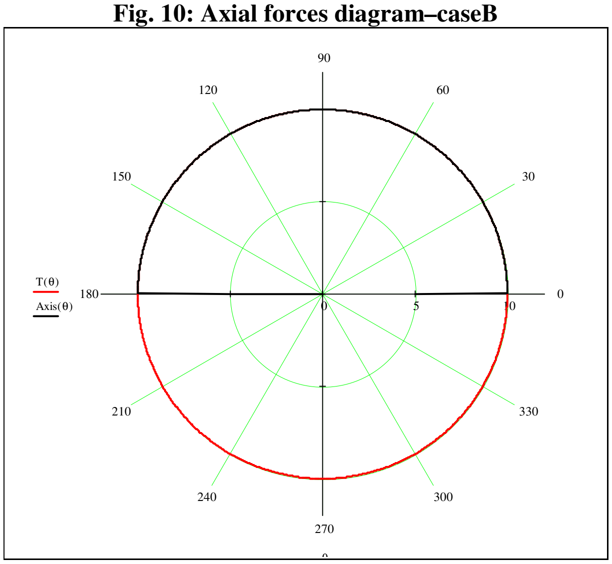

Fig.11: Shear forces diagram–case B

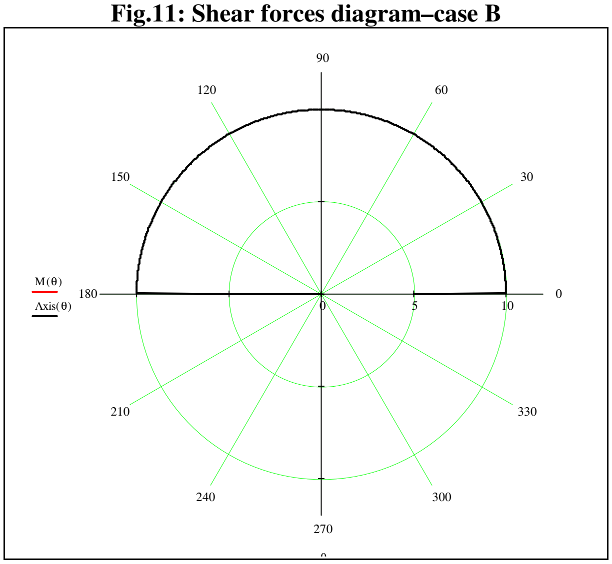

Fig. 12: Bending moments diagram–case B

Conclusion

The particular case B is a symmetrical one. The beam is loaded with one distributed radial load along its length. The diagrams from Fig. 10 – 12 show that the axial force are constant (negative) and shear forces are null along its length. The bending moment is also null along its length.

The vertical reaction in the fixed support V_B is equal to the force $Q=qR$ in section A and corresponds to 50% of the equivalent load q distributed along the semi-circle $Fe=2\ kN$.

The results are in perfect agreement with the experimental results and correspond to the expected behavior for a symmetrically loaded symmetric structure.

3. CONCLUSIONS

The following conclusions could be drawn by interpreting the numerical results of the particular cases:

– The method presented above allows the automated determination of the reaction forces, as function of the input parameters, as well as the plotting of the internal forces diagrams. The visualization of maximal and minimal values is as well enhanced.
– The polar coordinates diagrams allow the fast identification of the critical section(s) and the maximum value(s) of the bending moment for future verification / design of the beam.
– The method presented above has a high general character and can be verified by experimental data.

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